ANALYTICAL METHODS FOR INTRODUCTORY ECONOMICS

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INTRODUCTION

Introduction

- This review covers the essentials of **marginal analysis** and **linear functions** for students taking introductory courses in economics.
- Its primary focus is on the use and interpretation of **graphs**.

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Introduction

- It does not require calculus but some slides (marked with “*”) make a link to calculus for the benefit of students familiar with it.
- It also uses the language of calculus in some instances as a way to provide students with some exposure to mathematical notation and its value in expressing economic ideas with clarity and precision.

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PART 1: MARGINAL ANALYSIS

Marginal Analysis

- Consider an activity that can be undertaken in variable amounts (as opposed to binary activities that are "all or nothing").
  - Example: household electricity consumption.
- We will first describe the **marginal benefit** and **marginal cost** of this activity, and then bring these two concepts together to characterize optimal behaviour.

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Marginal Analysis

- The goal here is to understand how we can describe optimizing behaviour using simple marginal cost and marginal benefit curves, and why we can measure total costs and benefits as areas under these curves.

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1.1 MARGINAL BENEFIT

Marginal Benefit

- Suppose initially that the activity can be undertaken only in <u>discrete amounts</u>.
- Example:
  - electricity can be purchased only in integer <u>blocks of 100 kWh</u> (enough to run a 100 watt light bulb for 1000 hours).

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Marginal Benefit

- Imagine for the moment that a household is allowed to purchase no more than one block of electricity per month.
- They would use this electricity for the services that are most valuable to them.
- Suppose their **willingness-to-pay** for this first block is 28 cents per kWh (for a total of $28 for the 100 kWh block).

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Marginal Benefit

- Note that willingness-to-pay (WTP) here implicitly means maximum WTP, and it reflects the preferences and income of the household.
- Let us plot this WTP in a bar graph, where the vertical height of the bar depicts WTP measured in cents per kWh, and the horizontal axis measures consumption.

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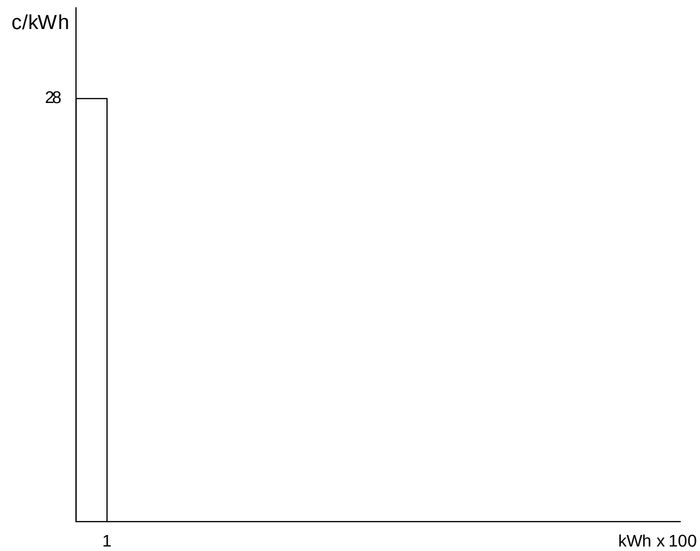

FIGURE A-1

15

Marginal Benefit

- Now suppose the household can purchase a second block of electricity per month.
- The first block is already allocated to the most important uses, so the second block will not be quite as valuable to the household.
- Suppose WTP for the second block is 26 cents per kWh (a total of $26 for the block).

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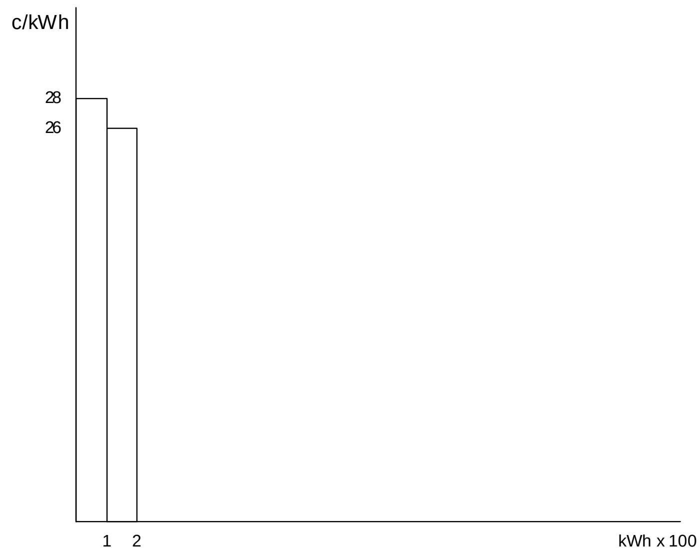

FIGURE A-2 17

Marginal Benefit

- Continue this valuation of additional blocks of electricity until WTP falls to zero (where additional electricity is of no value to the household at all).
- Suppose the resulting WTP schedule is as follows.

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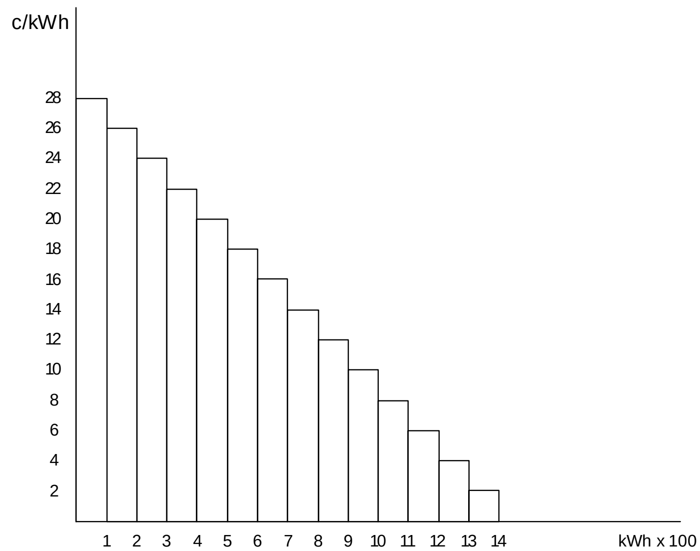

FIGURE A-3

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Marginal Benefit

- In the example illustrated, WTP falls linearly (at a constant rate of 2 cents per kWh for each additional block) but it could of course fall according to a different pattern, depending on household preferences and income.
- The particular pattern of decline is unimportant for our purposes.

20

Marginal Benefit

- The vertical heights depicted in Figure A-3 are **marginal benefit** values for the household:
  - they measure the benefit per kWh (as measured by WTP) from each incremental block of electricity.

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Marginal Benefit

- We can use this marginal benefit data to construct the **total benefit** to the household from any given level of electricity consumption.
- This total benefit is represented as an **area** on our marginal benefit graph.
- For the first block consumed:
  - total benefit = 28 c/kWh x 100 kWh = $28

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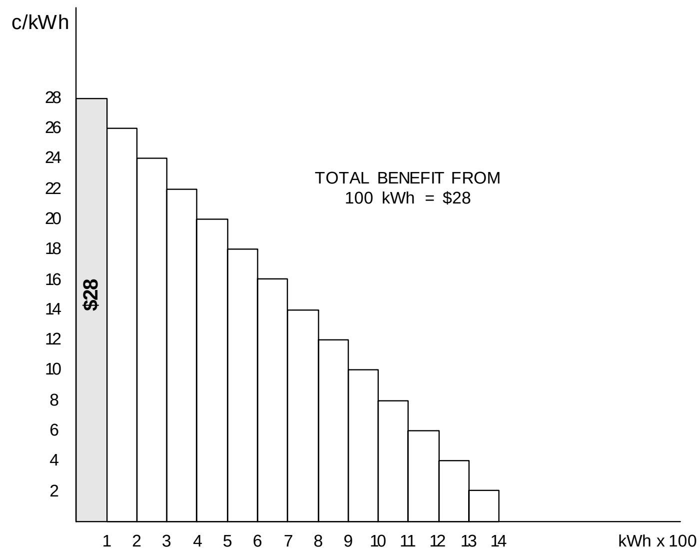

FIGURE A-4

23

Marginal Benefit

- Note that **units of measure** are important:
  - Marginal benefit is measured as a rate (in units of money <u>per</u> kWh) while total benefit is measured as an absolute (in units of money)
  - Thus, the calculation of total benefit is

$$28\frac{\text{cents}}{\text{kWh}}*100\text{kWh} = 2800\text{cents} = \$28$$

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Marginal Benefit

- Similarly, total benefit from the second block is calculated as

$$26\frac{\text{cents}}{\text{kWh}}*100\text{kWh}=2600\text{cents}=\$26$$

- Thus, from the first two blocks combined:
  - total benefit = $28 + $26 = $54

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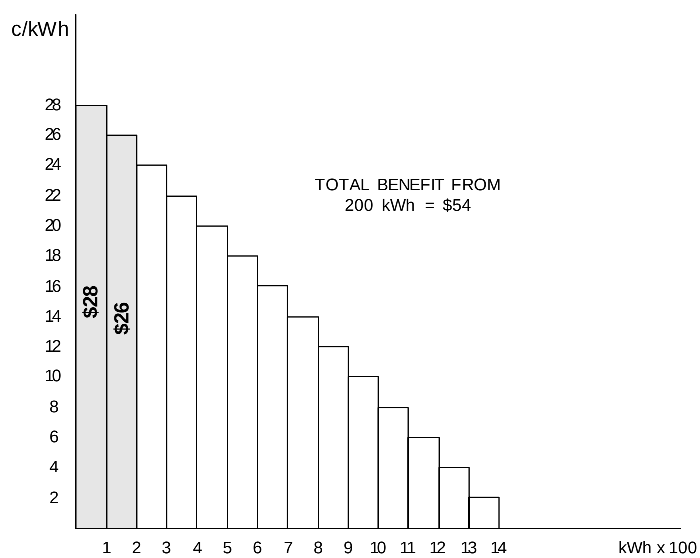

FIGURE A-5

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Marginal Benefit

- From the first three blocks combined:
  - total benefit = $54 + $24 = $78

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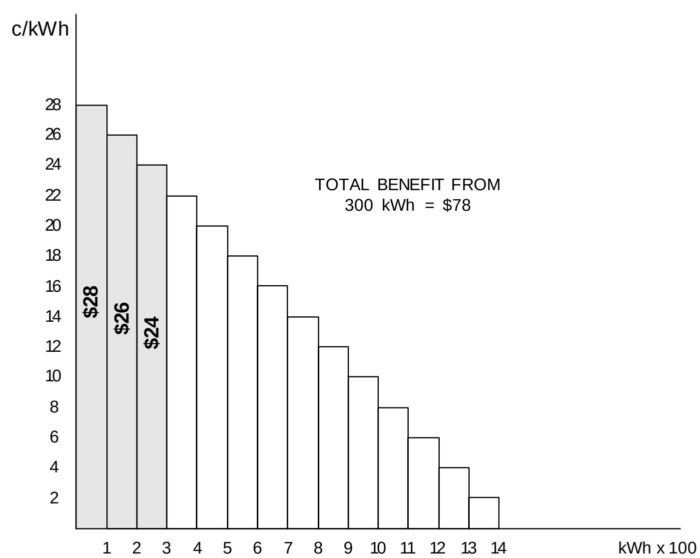

FIGURE A-6

28

Marginal Benefit

- Following this procedure, we can calculate the area representing the cumulative benefit as consumption rises until additional consumption yields no additional benefit.
- At 1400 kWh, the total benefit from consumption is $210; see Figure A-7.

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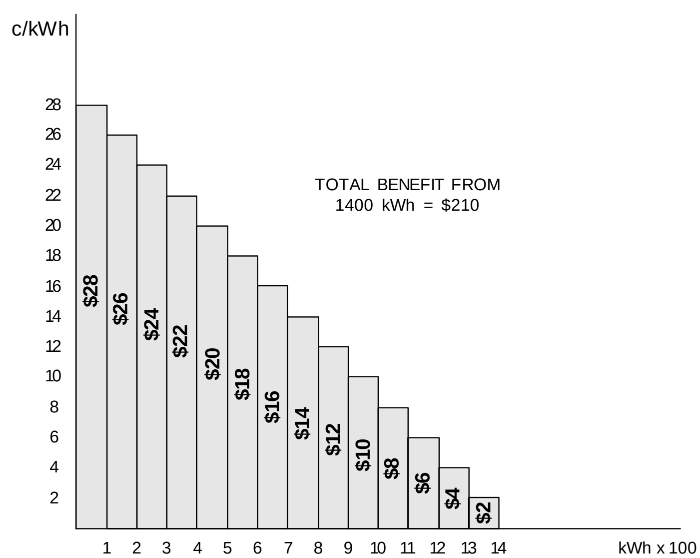

FIGURE A-7

30

Marginal Benefit

- We can write this calculation of total benefit from 14 blocks consumed as

$$B_{14} = b_1 + b_2 + b_3 + \ldots + b_{14} = \sum_{i=1}^{14} b_i$$

where b_1 = \$28 is the total benefit from the first block, b_2 = \$26 is the total benefit from the second block, etc.

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Marginal Benefit

- In general, the total benefit from x blocks consumed is

$$B_x = \sum_{i=1}^{x} b_i$$

32

Marginal Benefit

- An <u>alternative way</u> to present the total benefit data is in a separate bar graph, where the vertical height of a bar measures the total benefit from the given number of blocks consumed.
- See Figure A-8, where vertical axis measures dollars of benefit.

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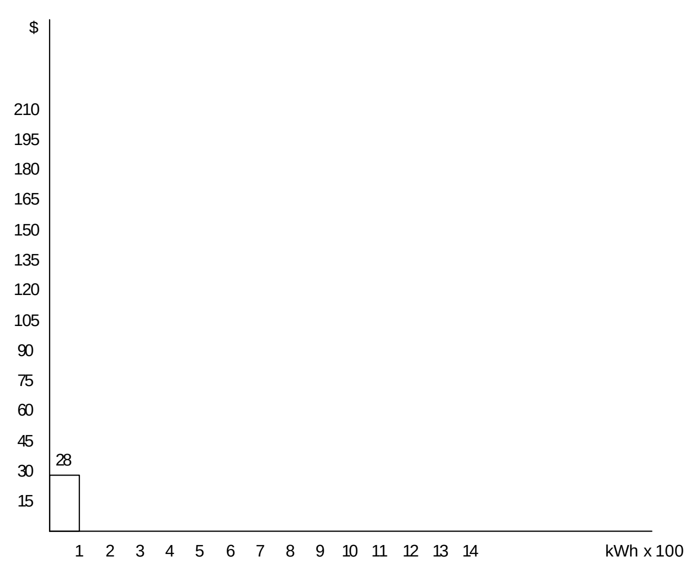

FIGURE A-8

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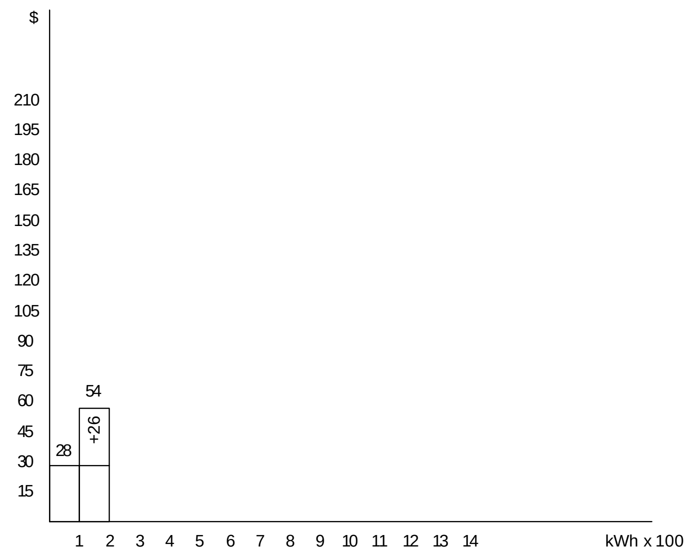

FIGURE A-9 35

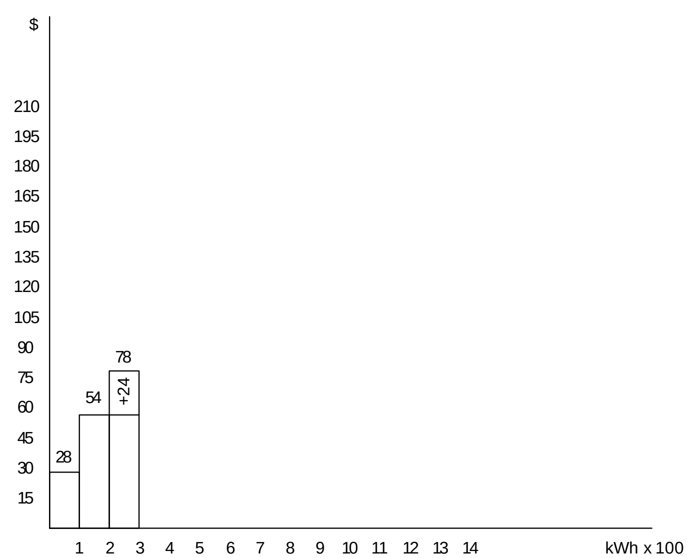

FIGURE A-10 36

Marginal Benefit

- Proceeding in this way yields the **total benefit schedule**; see Figure A-11.

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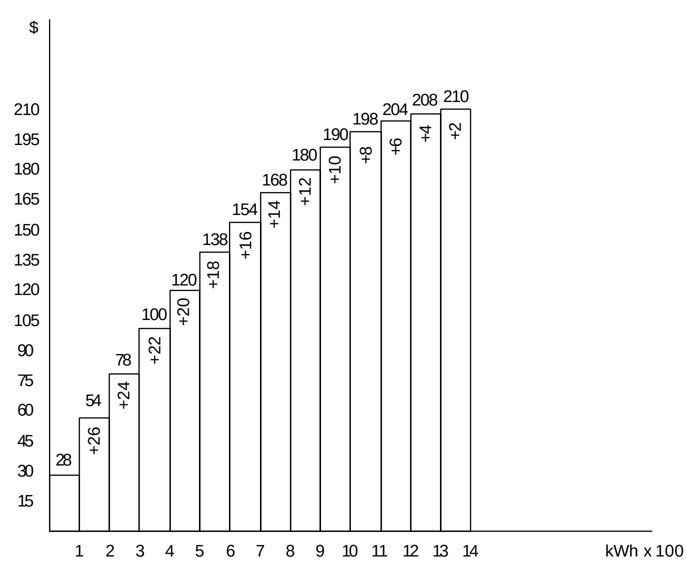

FIGURE A-11

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Marginal Benefit

- Note that the height of the total benefit graph at 14 units in Figure A-11 is equal to the area we calculated from the marginal benefit graph in Figure A-7:

$$B_{14} = \sum_{i=1}^{14} b_i = 210$$

- In both cases we are adding-up the benefit from each block to calculate total benefit.

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Marginal Benefit Using Continuous Variables

- Now suppose that electricity can be purchased in infinitesimally small amounts (tiny fractions of a kWh) rather than discrete blocks of 100 kWh each.
- We can then effectively treat electricity as a continuous variable.

40

Marginal Benefit

- Using a continuous variable also allows us to use calculus, though a knowledge of calculus is not needed here.
- Slides marked with "*" introduce calculus and are intended only for those students with a prior knowledge of calculus.
- Other students can simply skip these slides.

41

Marginal Benefit

- Consider the transformation of the marginal benefit schedule as we move from the discrete variable framework to a framework with a continuous variable.
- As a starting point, let us suppose that electricity can now be purchased in half-blocks (each comprising 50 kWh).

42

Marginal Benefit

- The WTP for any given block of 100kWh is unchanged but within that block, the WTP for the first half will be greater than WTP for the second half.

43

Marginal Benefit

- Why? The explanation is the same as for why the first full block is more highly valued than the second full block:
  - electricity is allocated by the household to its various uses in order of importance, and hence value.

44

Marginal Benefit

- Recall that WTP for the first block of 100kWh is 28 cents per kWh.
- We can now think of that as the <u>average</u> of the WTP per kWh for the first half-block and the WTP per kWh for the second half-block.

45

Marginal Benefit

- For example, these half-block WTP values might be 28.5 cents per kWh for the first half-block and 27.5 cents per kWh for the second half-block, for an average of 28 cents per kWh for the block as a whole:

$$\frac{28.5+27.5}{2}=28$$

46

Marginal Benefit

- Graphically, we are splitting the total benefit from the first block into two areas that sum to $28:

$$\begin{gathered}28.5\text{c/kWh} * 50\text{kWh} \\ + \\ 27.5\text{c/kWh} * 50\text{kWh}\end{gathered}$$

$$= \$14.25 + \$13.75$$
$$= \$28$$

47

Marginal Benefit

- Graphically, we are splitting the total benefit from the first block into two areas that sum to $28:

$$28.5\text{c/kWh} * 50\text{kWh} \; + \; 27.5\text{c/kWh} * 50\text{kWh}$$

$$= \$14.25 + \$13.75$$
$$= \$28$$

- See Figure A-12

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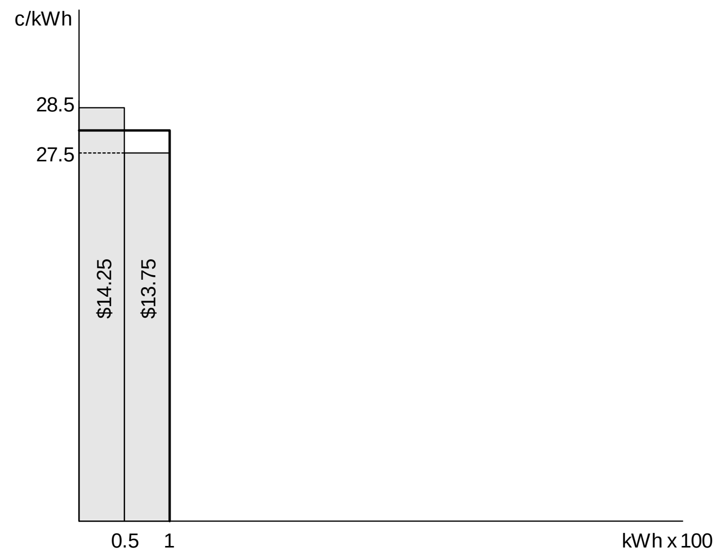

FIGURE A-12

49

Marginal Benefit

- We can make the same transformation to the other blocks, splitting each block into two halves, where the WTP for the first half will be greater than the WTP for the second half, but the total for the block as a whole is unchanged.
- For example, splitting the second block yields two areas that sum to $26.

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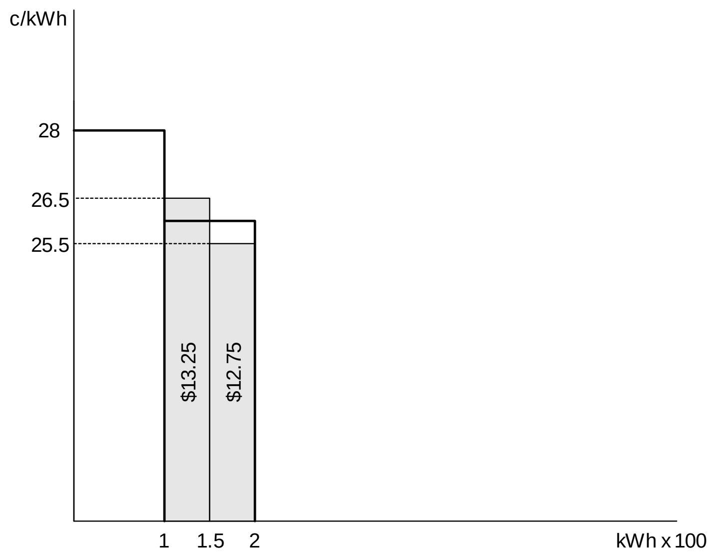

FIGURE A-13

51

Marginal Benefit

- Let us now take one step further towards thinking in terms of a continuous variable by splitting the first block into four quarters of 25kWh each.

52

Marginal Benefit

- The average of 28 cents per kWh for that first block might be made up:

  28.75 c/kWh for the first 25kWh

  28.25 c/kWh for the next 25 kWh

  27.75 c/kWh for the next 25 kWh

  27.25 c/kWh for the next 25 kWh

53

Marginal Benefit

- Confirm that the average WTP across these four quarters is still 28 c/kWh:

$$\frac{28.75 + 28.25 + 27.75 + 27.25}{4} = 28$$

54

Marginal Benefit

- The total WTP for the block is still \$28; we have simply divided the benefit from that first block into four areas that sum to \$28:

$$28.75\text{c/kWh} * 25\text{kWh} = \$7.10$$
$$28.25\text{c/kWh} * 25\text{kWh} = \$7.06$$
$$27.75\text{c/kWh} * 25\text{kWh} = \$6.94$$
$$27.25\text{c/kWh} * 25\text{kWh} = \$6.81$$

55

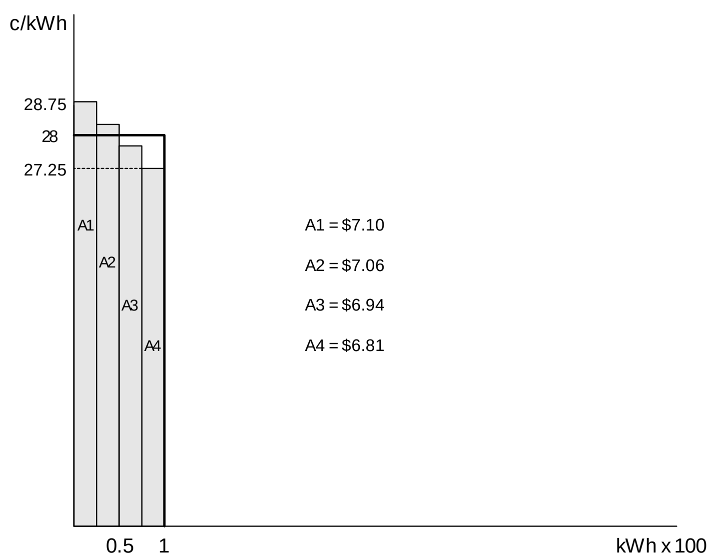

FIGURE A-14

56

Marginal Benefit

- Again, we can do the same for the second and subsequent blocks.
- Moreover, we can continue to split the blocks increasingly finely until we effectively have a continuous variable for electricity consumption, and a continuous decline in WTP within any block.
- See Figure A-15 for the first block.

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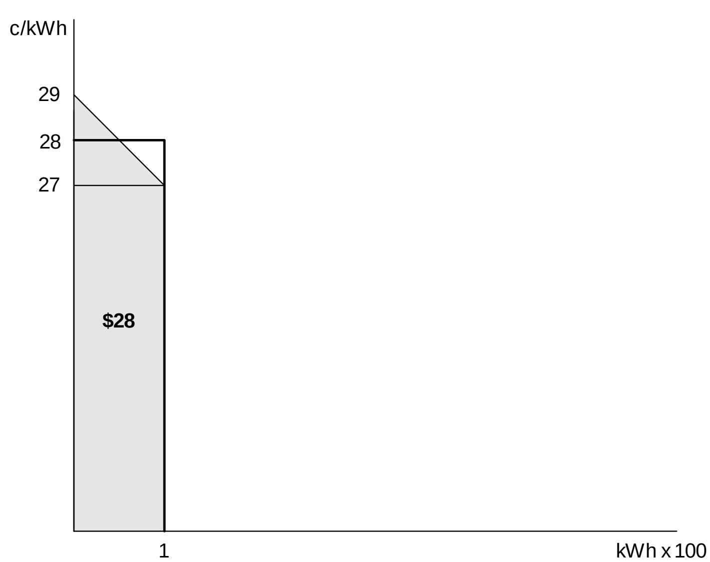

FIGURE A-15

58

Marginal Benefit

- The area under the continuous WTP schedule for the first 100 kWh is still $28:
  - the area of the lower rectangle in Figure A-15 (below the dashed line) is

$$1*27 = 27$$

  - the area of the upper triangle (above the dashed line) is

$$\tfrac{1}{2}(1*2) = 1$$

59

Marginal Benefit

- This combined area under the continuous graph can be thought of as the sum of an infinite number of infinitesimally small rectangular areas (akin to those in Figure A-14) as we split the first 100 kWh block into tiny fractions of a kWh.

60

Marginal Benefit

- Applying this same logic to the second and subsequent blocks allows us to represent the WTP data with a continuous WTP schedule.

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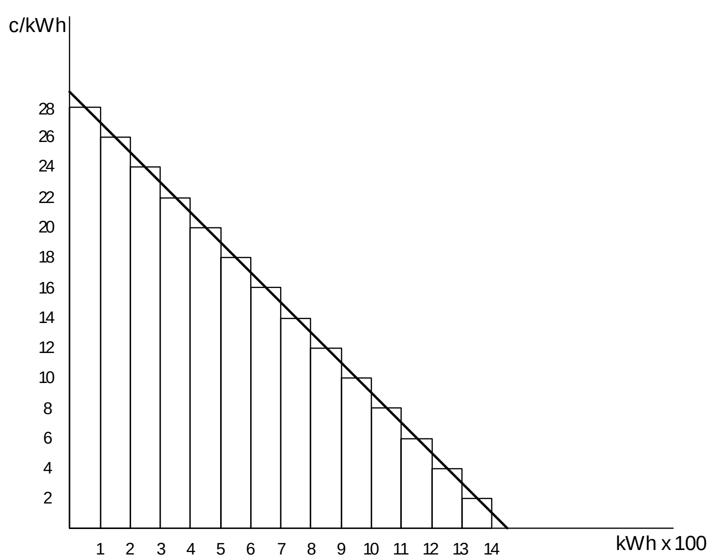

FIGURE A-16

62

Marginal Benefit

- If the household is not constrained to purchase discrete blocks, and may instead purchase electricity in any amount, then we can dispense with the blocks entirely.
- We can simply graph WTP as a continuous **marginal benefit schedule**, as in Figure A-17.

63

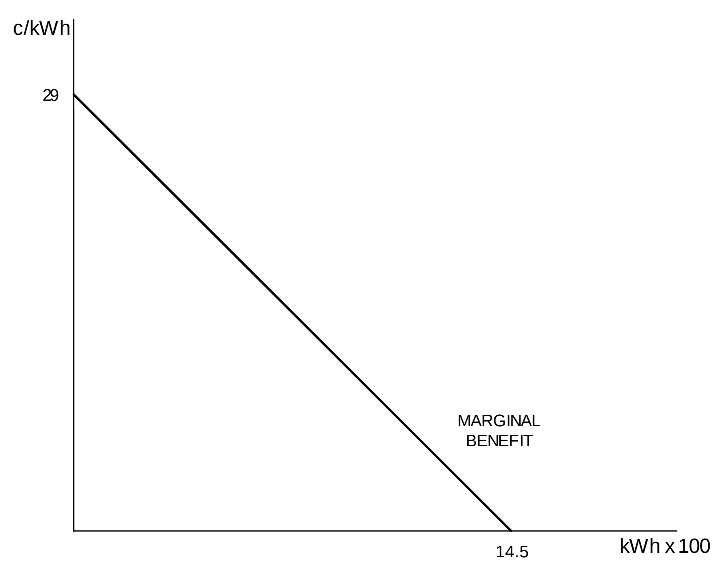

FIGURE A-17

64

Marginal Benefit

- Let us now summarize and formalize the interpretation of this continuous marginal benefit schedule.
- Let z denote the consumption of electricity (where the unit of measure is kWh).
- Let $MB(z)$ denote the marginal benefit from electricity consumption, measured in \$ per unit.

65

Marginal Benefit

- Suppose the household is currently consuming z^0 kWh of electricity.
- The total benefit from this consumption is the area under the $MB(z)$ schedule between zero and z^0, and is denoted $B(z^0)$.
- See Figure A-18.

66

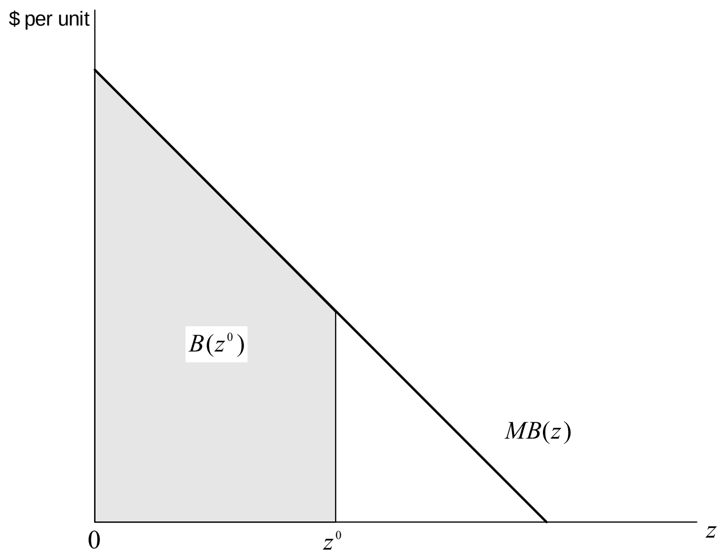

FIGURE A-18

67

Marginal Benefit

- This area is called the **definite integral** of the marginal benefit function between zero and z^0, and it is written as

$$B(z^0) = \int_0^{z^0} MB(z)dz$$

68

Marginal Benefit

- This benefit measure is the maximum that the household would be willing to pay for z^0 kWh of electricity.
- Their WTP for any additional amount – call it Δz – is the area under the $MB(z)$ schedule between z^0 and $z^0 + \Delta z$, labeled ΔB in Figure A-19.

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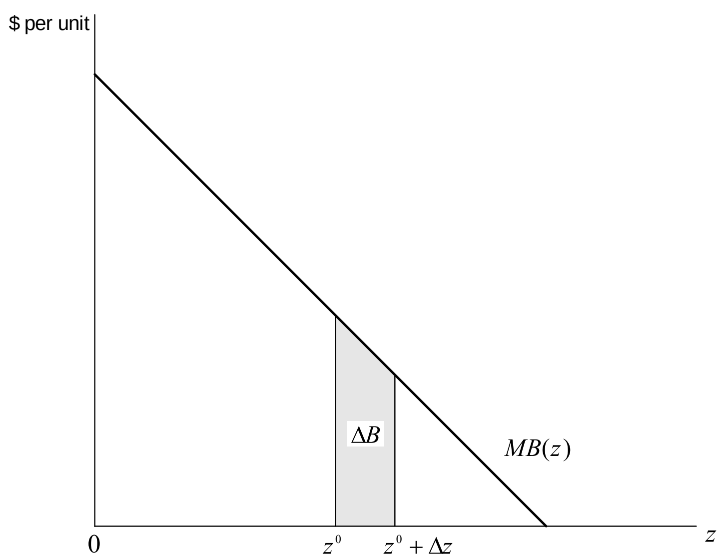

FIGURE A-19

70

Marginal Benefit

- This area is called the definite integral of the marginal benefit function between z^0 and $z^0 + \Delta z$, and it is written as

$$\Delta B\Big|_{z=z^0} = \int_{z^0}^{z^0+\Delta z} MB(z)dz$$

where the vertical bar means "evaluated at".

71

Marginal Benefit

- Let us use our discrete-variable WTP data to construct a numerical example.
- The functional form that fits our data is

$$MB(z) = 29 - 2z$$

where MB is measured in cents per kWh, and z is measured in kWh x 100.
- See Figure A-20.

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FIGURE A-20

73

Marginal Benefit

- Suppose the household currently consumes exactly 700 kWh of electricity (seven of our previous discrete blocks).
- The total benefit from this consumption can be calculated as the sum of the two areas in Figure A-21.

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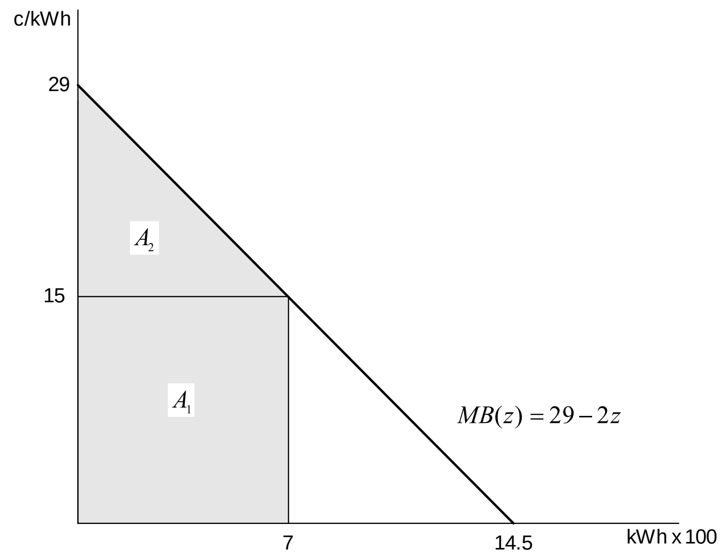

FIGURE A-21

75

Marginal Benefit

- The total benefit (measured in $) is:

$$A_1 + A_2 = 15 \text{ x } 7 + \frac{14 \text{ x } 7}{2} = 154$$

76

Marginal Benefit

- Now compare this value with the data from our discrete-value total benefit schedule.

77

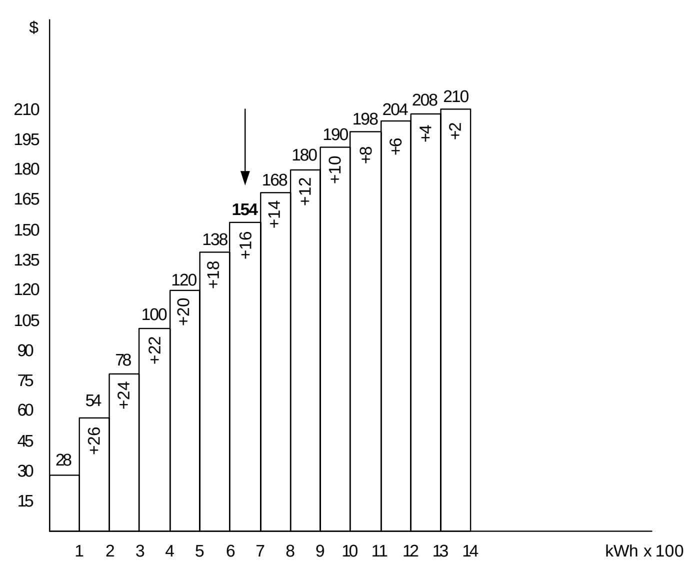

FIGURE A-22

78

Marginal Benefit

- This relationship between the area under the continuous $MB(z)$ schedule and the total benefit schedule holds at every level of consumption in the discrete data.
- Exercise: show that it holds at 1000 kWh.

79

*** Marginal Benefit ***

- **If you know calculus**, compare this area with the evaluation of the definite integral.
- The definite integral of interest is

$$B(7) = \int_0^7 (29 - 2z)dz$$

80

*** Marginal Benefit ***

- Evaluating this integral yields

$$B(7) = \left[29z - z^2\right]_0^7 = \left[29(7) - (7)^2\right] - 0 = 154$$

- This is necessarily the same value as the area we calculated from Figure A-21.

81

Marginal Benefit

- Similarly, we can calculate the <u>additional</u> benefit from increasing consumption from 700 kWh to 800 kWh.

- See Figure A-23.

82

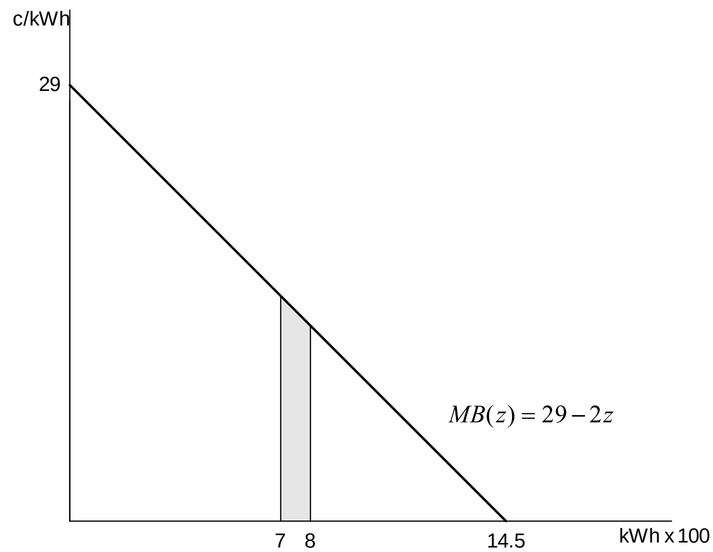

FIGURE A-23

83

Marginal Benefit

- This area is the definite integral:

$$\Delta B\Big|_{z=7} = \int_7^8 (29 - 2z)\,dz$$

and we can calculate it from our graph as the sum of the two areas in Figure A-24.

84

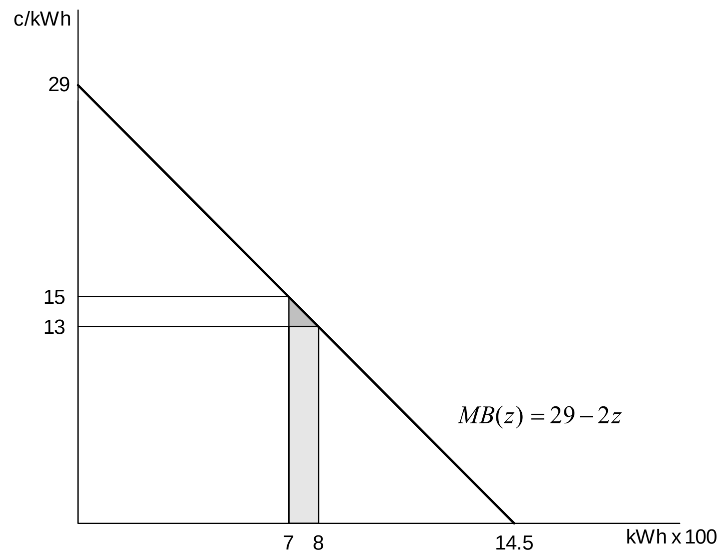

FIGURE A-24

85

Marginal Benefit

- The area of the shaded triangle (measured in \$) is

$$\frac{2 \text{ x } 1}{2} = 1$$

- The area of the shaded rectangle is \$13.
- Hence, the total area is \$14.

86

Marginal Benefit

- Now compare this with the marginal benefit of the 8th unit from our discrete WTP data.

87

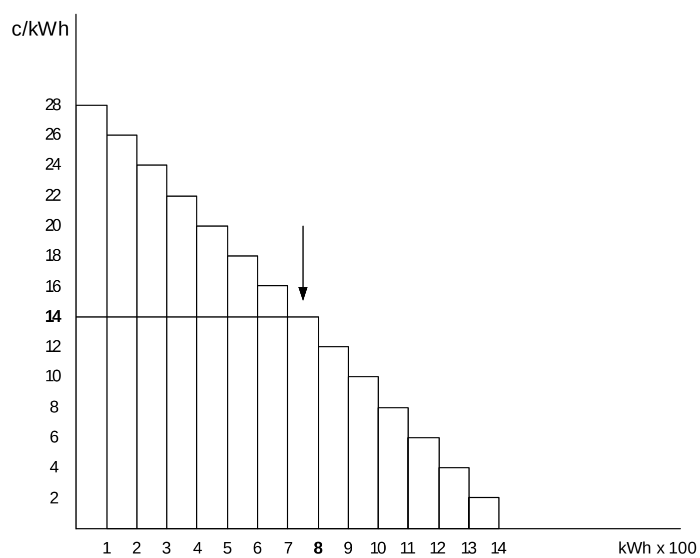

FIGURE A-25

88

Marginal Benefit

- Again, this relationship between our continuous $MB(z)$ schedule and our discrete WTP data holds at any level of consumption.
- Exercise: show that it holds for an increase in consumption from 300 kWh to 400 kWh.

89

*** Marginal Benefit ***

- **If you know calculus**, evaluate the integral on slide 79 to yield

$$\begin{aligned}\Delta B &= \left[29z - z^2\right]_7^8 \\ &= \left[29(8) - (8)^2\right] - \left[29(7) - (7)^2\right] \\ &= 14\end{aligned}$$

90

Marginal Benefit

- We have so far constructed a continuous representation of the marginal benefit schedule.
- It is also useful to construct a continuous representation of the total benefit schedule.
- To begin, recall our total benefit schedule for the discrete data.

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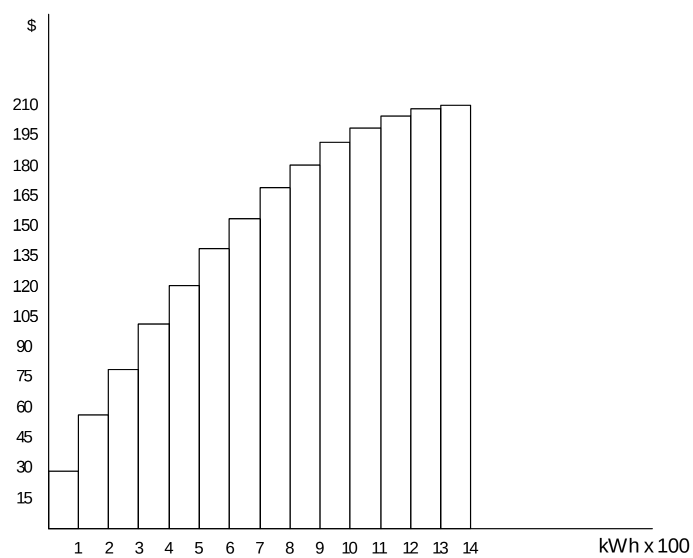

FIGURE A-26

92

Marginal Benefit

- Now fit a continuous curve that interpolates values between the integer consumption levels to construct a continuous total benefit function.
- See Figure A-27.

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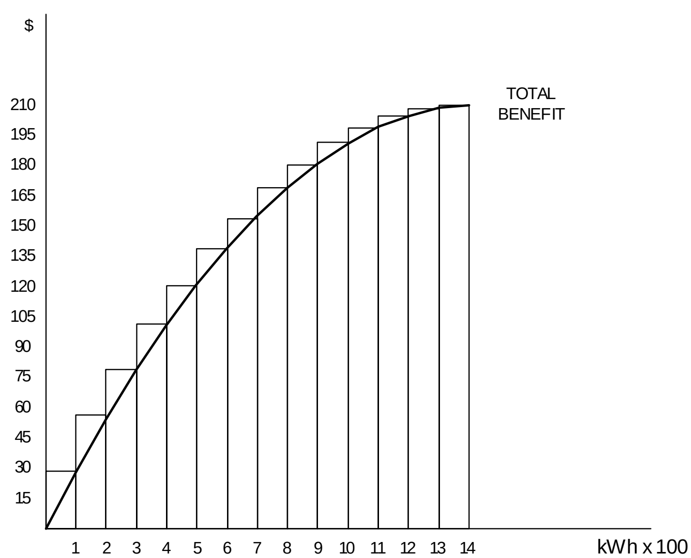

FIGURE A-27

94

Marginal Benefit

- Let us conduct this interpolation exercise a bit more rigorously.
- Recall that we constructed the total benefit schedule from our discrete data by adding-up the benefits from each block consumed.

95

Marginal Benefit

- In particular, the total benefit from *x* units was calculated as

$$B_x = \sum_{i=1}^{x} b_i$$

96

Marginal Benefit

- The continuous-variable analogue of "adding-up" is integration (that is, taking an area).
- Thus, the value of the total benefit function at any given consumption level z^0 is simply

$$B(z^0) = \int_0^{z^0} MB(z)dz$$

97

Marginal Benefit

- By calculating this integral (which is just an area) at every possible value of z^0 and plotting the results, we can derive the total benefit function, denoted $B(z)$.
- See Figure A-28.

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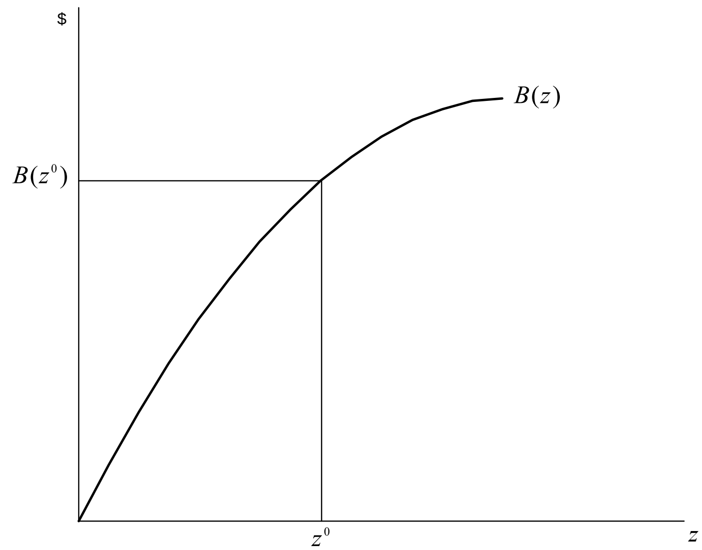

FIGURE A-28 99

*** Marginal Benefit ***

- **If you know calculus**, recall from our example data that

$$MB(z) = 29 - 2z$$

- Hence, the total benefit when $z = x$ is

$$B(x) = \int_0^x (29 - 2z)dz = 29x - x^2$$

100

***Marginal Benefit ***

- Since x is just a particular value of z, and could potentially be any value of z, we typically just use z in place of x when we write the total benefit function:

$$B(z) = 29z - z^2$$

- See Figure A-29.

101

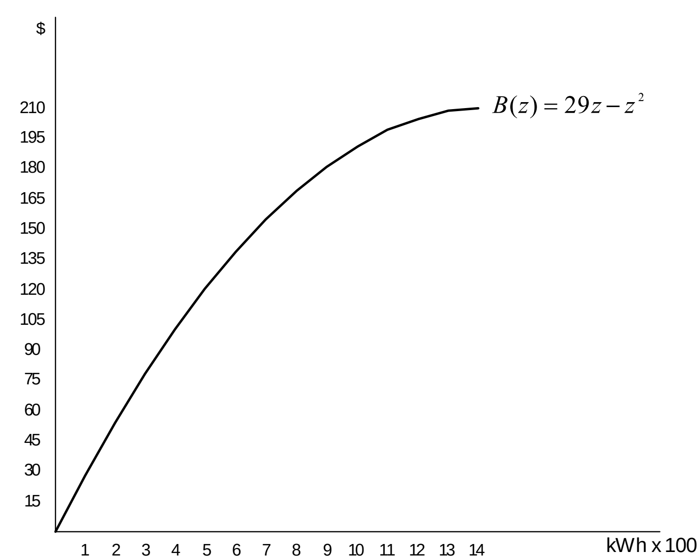

FIGURE A-29 *

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Marginal Benefit

- To summarize, we construct the total benefit schedule by integrating (taking an area) under the marginal benefit schedule.

103

Marginal Benefit

- We can also go in the reverse direction:
  - we can construct the marginal benefit function from the total benefit function by calculating its slope.

104

Marginal Benefit

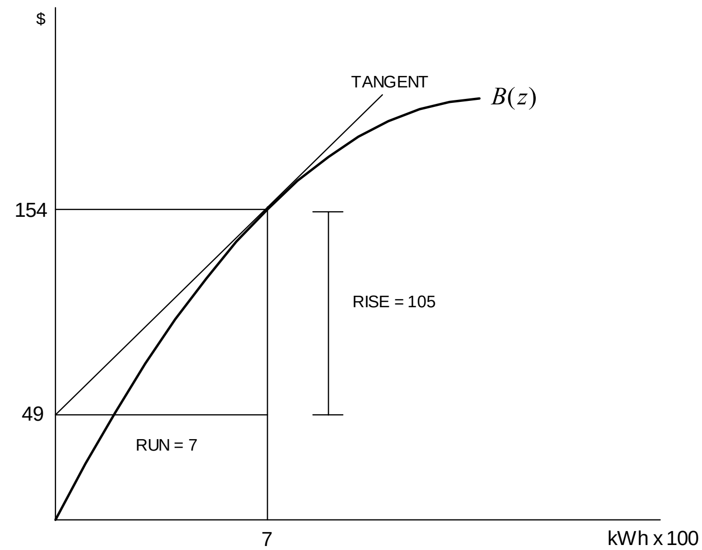
- In particular, the marginal benefit function measures the **rate of change** (or slope) of the total benefit function.
- To find the rate of change of a non-linear continuous function at a particular point, we calculate the slope of a tangent to the function at that point.

105

Marginal Benefit

- For example, what is the rate of change of the total benefit function at 700 kWh?
- To calculate this exactly we would need to derive the specific continuous function representing total benefit.
- This requires calculus so we will not do it here, but Figure A-30 illustrates the result we would obtain.

106



FGIURE A-30

107

Marginal Benefit

- From Figure A-30, the slope of the tangent at 700 kWh is

$$\frac{rise}{run} = \frac{105}{7} = 15$$

where this ratio is measured in c/kWh (or equivalently, $/100 kWh).

108

Marginal Benefit

- Now compare this with our $MB(z)$ schedule at 700 kWh.

109

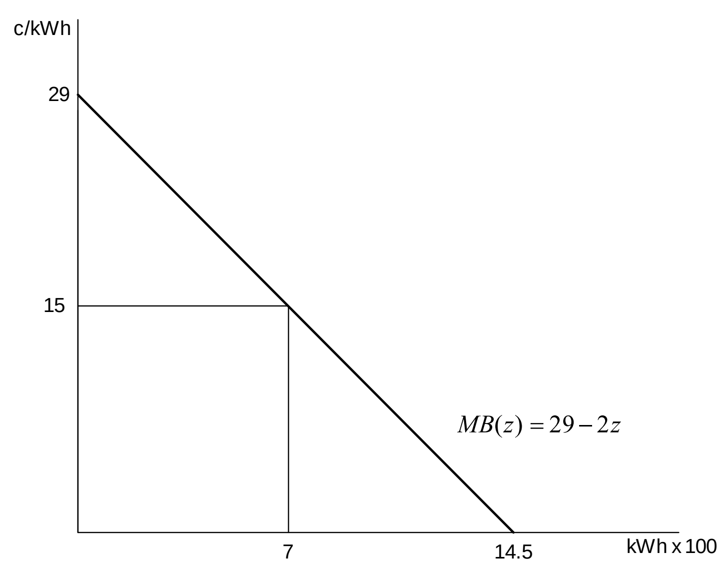


FIGURE A-31

110

Marginal Benefit

- This relationship between the slope of the total benefit function and the marginal benefit function holds at every value of z.

111

Marginal Benefit

- To summarize, the marginal benefit function measures the rate of change (or slope) of the total benefit function, and total benefit is measured as an area under the marginal benefit function.

112

*** Marginal Benefit ***

- **If you know calculus**, the marginal benefit function is the **first derivative** of the total benefit function:

$$MB(z) = \frac{dB(z)}{dz}$$

113

*** Marginal Benefit ***

- Recall that the total benefit function is

$$B(z) = 29z - z^2$$

- Differentiation with respect to z yields

$$MB(z) = 29 - 2z$$

114

*** Marginal Benefit ***

- This reflects the fact that differentiation is essentially the "reverse" of integration, and we constructed $B(z)$ by integrating $MB(z)$.
- Thus,

$$\frac{dB(z)}{dz} = \frac{d\left(\int MB(z)dz\right)}{dz} = MB(z)$$

115

1.2 MARGINAL COST

Marginal Cost

- We have so far focused exclusively on the benefits from an activity.
- Choices about how much of an activity to undertake involve balancing benefits and costs, and it is in the study of these choices that marginal analysis is most useful.

117

Marginal Cost

- Suppose our household cannot purchase electricity on the market but must instead generate their own electricity (by spinning a turbine by hand, for example).
- This is time-consuming and tiring, and hence costly.

118

Marginal Cost

- In particular, time and effort devoted to generating electricity must be taken away from some other valuable activity, and hence it has an opportunity cost.
- In general, the **opportunity cost** of an activity measures the value of the <u>next best</u> alternative use of the resources used in that activity.

119

Marginal Cost

- The **marginal cost** (MC) of an activity measures the opportunity cost of undertaking <u>one more unit</u> of that activity.

120

Marginal Cost

- In the context of our electricity example, suppose the continuous marginal cost schedule for generating electricity is as depicted in Figure A-32, where *z* denotes electricity production in kWh.

121

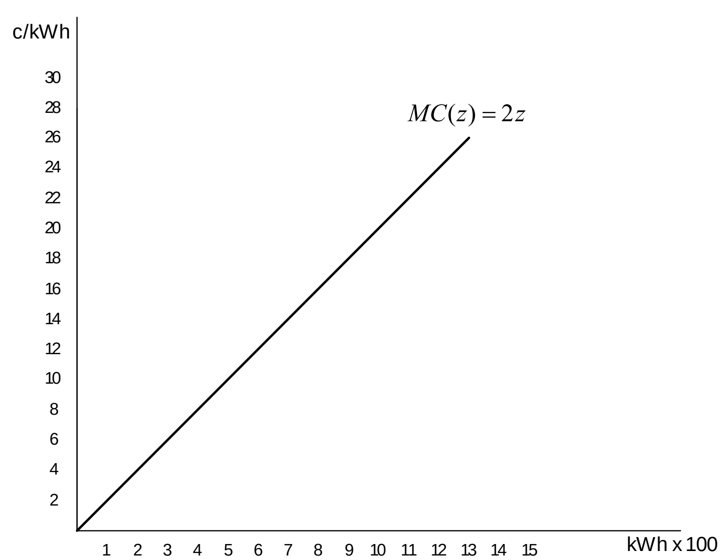


FIGURE A-32

122

Marginal Cost

- Why is this MC schedule upward-sloping?
  – if the household behaves rationally, then the time devoted to generating the first units of electricity are taken away from the least valuable alternative uses of that time, so the cost is relatively low.
  – but as more electricity is produced, the required time must be taken away from increasingly valuable alternative uses.

123

Marginal Cost

- The particular shape (and intercept) of the *MC*(*z*) illustrated in Figure A-32 are not especially important for our purposes.
- The critical feature is its positive slope.

124

Marginal Cost

- The **total cost** of generating a given amount of electricity (excluding fixed costs, which are independent of the quantity generated) is calculated as an area under the *MC*(z) schedule.
- The logic is precisely the same as that underlying our calculation of total benefit.

125

Marginal Cost

- For example, the total cost of producing 900 kWh is $81, as illustrated in Figure A-33 .

126

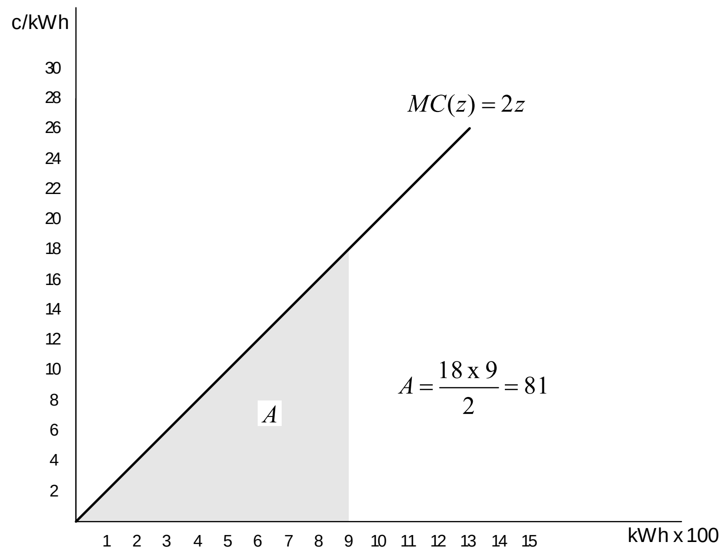


FIGURE A-33

127

Marginal Cost

- In general, the total cost in dollars of producing z^0 units is

$$C(z^0) = \int_0^{z^0} MC(z)dz$$

where MC is measured in dollars per unit.

128

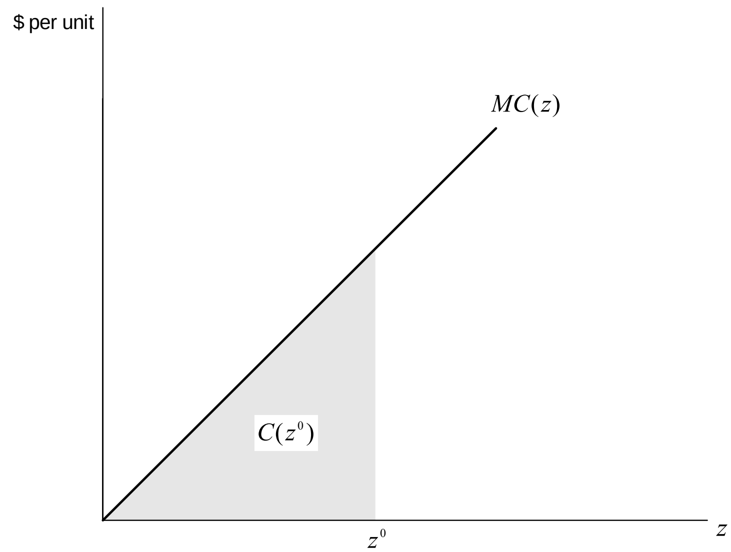


FIGURE A-34

129

Marginal Cost

- We can also construct a separate **total cost schedule** by plotting $C(z^0)$ at every possible value of z^0.
- For the particular example, the total cost schedule looks as follows.

130

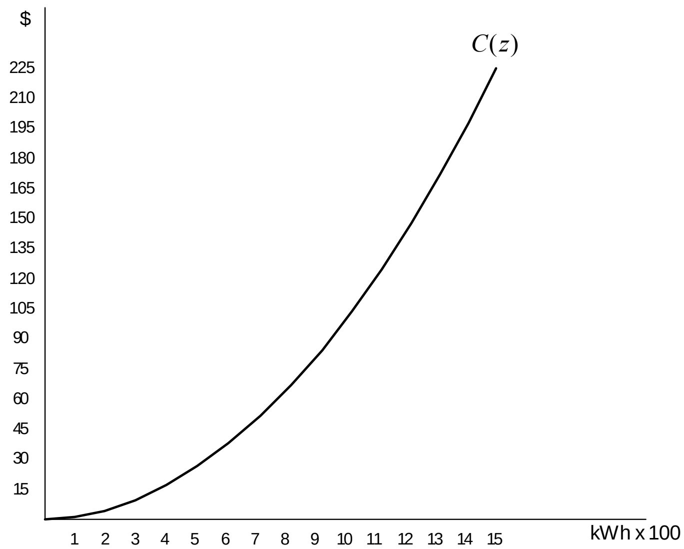


FIGURE A-35

131

Marginal Cost

- The marginal cost function measures the **rate of change** (or slope) of this total cost function.
- Recall that to find the rate of change of a continuous function at a particular point, we calculate the slope of a tangent to the function at that point.

132

Marginal Cost

- For example, what is the rate of change of total cost at 1000 kWh?
- To calculate this exactly we would need to derive the specific continuous function representing total cost.
- This requires calculus so we will not do it here, but Figure A-36 illustrates the result we would obtain.

133

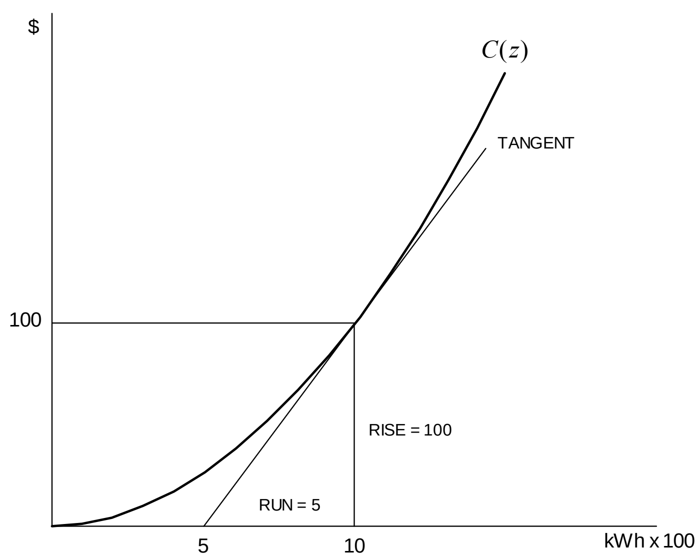


FIGURE A-36

134

*** Marginal Cost ***

- **If you know calculus**, $C(z)$ is constructed by evaluating the indefinite integral

$$C(z) = \int (2z)dz = z^2$$

135

Marginal Cost

- The slope of the tangent at 1000 kWh is

$$\frac{rise}{run} = \frac{100}{5} = 20$$

where this ratio is measured in c/kWh, or equivalently, \$/100 kWh.

136

Marginal Cost

- Now compare this with our $MC(z)$ schedule at 1000 kWh.

137

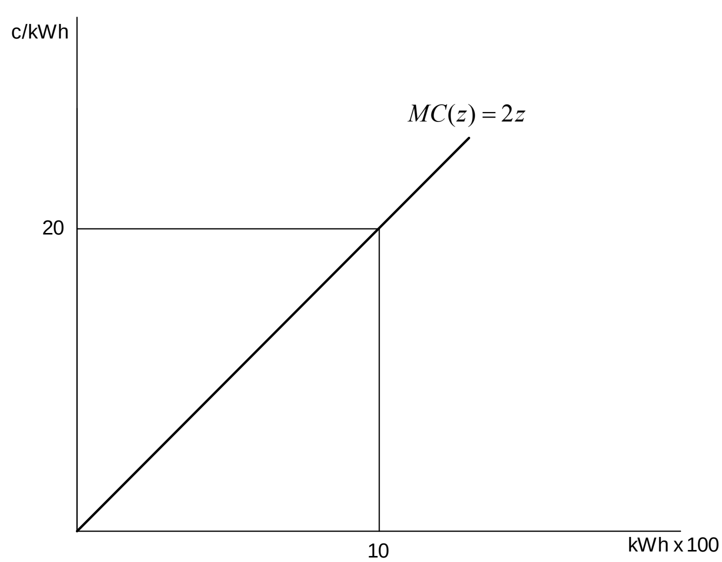


FIGURE A-37

138

Marginal Cost

- This relationship between the slope of the total cost function and the marginal cost function holds at every value of z.

139

Marginal Cost

- To summarize, the marginal cost function measures the rate of change (or slope) of the total cost function, and total cost is measured as an area under the marginal cost function.

140

*** Marginal Cost ***

- **If you know calculus**, the marginal cost function is the **first derivative** of the total cost function:

$$MC(z) = \frac{dC(z)}{dz}$$

141

*** Marginal Cost ***

- Recall that the total cost function is

$$C(z) = z^2$$

- Differentiation with respect to z yields

$$MC(z) = 2z$$

142

1.3 OPTIMIZATION

Optimization

- We now want to use marginal analysis to characterize optimal decision-making.
- In particular, how much electricity should our household generate?

144

Optimization

- Our **behavioral assumption** is that the household acts to maximize the **net benefit** from the activity, where net benefit is the difference between total benefit and total cost:

$$NB(z) = B(z) - C(z)$$

145

Optimization

- In graphical terms, $NB(z)$ is the vertical distance between $B(z)$ and $C(z)$.
- See Figure A-38.

146

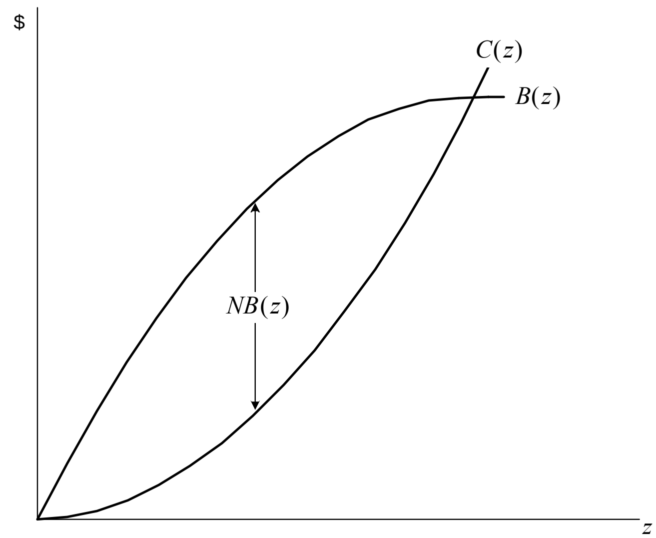


FIGURE A-38

147

Optimization

- At what level of z is $NB(z)$ <u>maximized</u>?
- We can solve this problem using marginal analysis, as follows.

148

Optimization

- Suppose the household is currently producing z^0 units of electricity, with an associated cost of $C(z^0)$ and an associated benefit of $B(z^0)$.

149

Optimization

- Recall that these cost and benefit values are measured as areas under the $MC(z)$ and $MB(z)$ schedules respectively; see Figures A-39 and A-40.

150

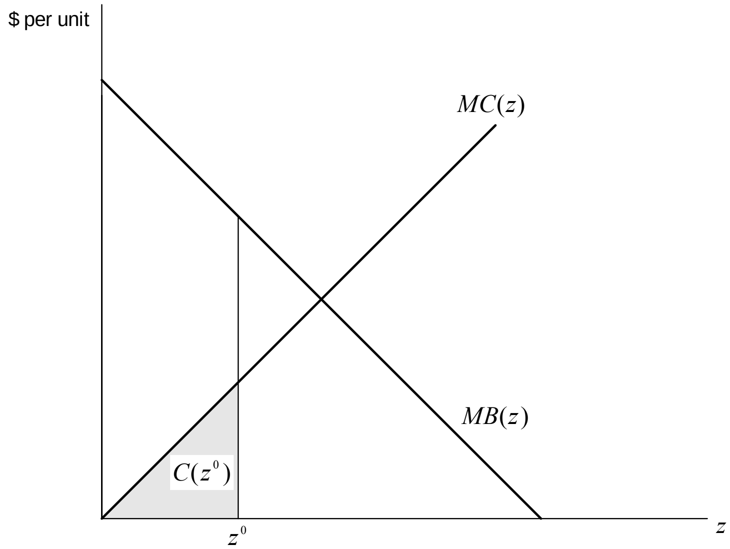


FIGURE A-39 151

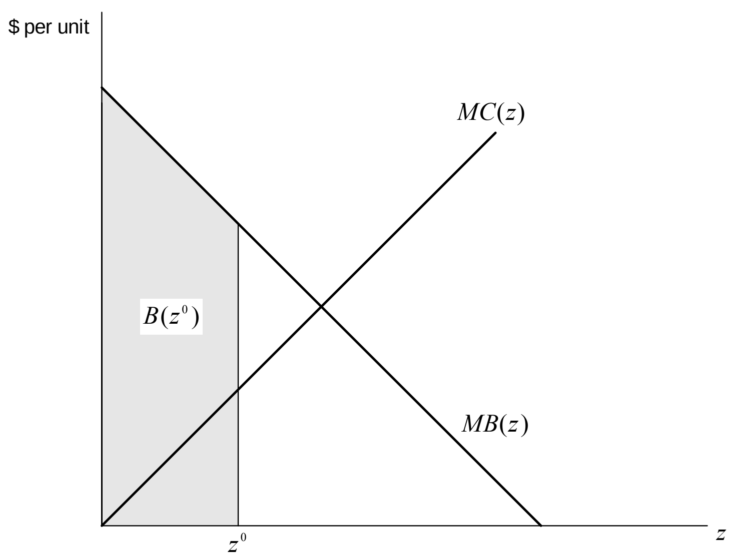


FIGURE A-40 152

Optimization

- The net benefit at z^0 is the difference between these two areas; see Figure A-41.

153

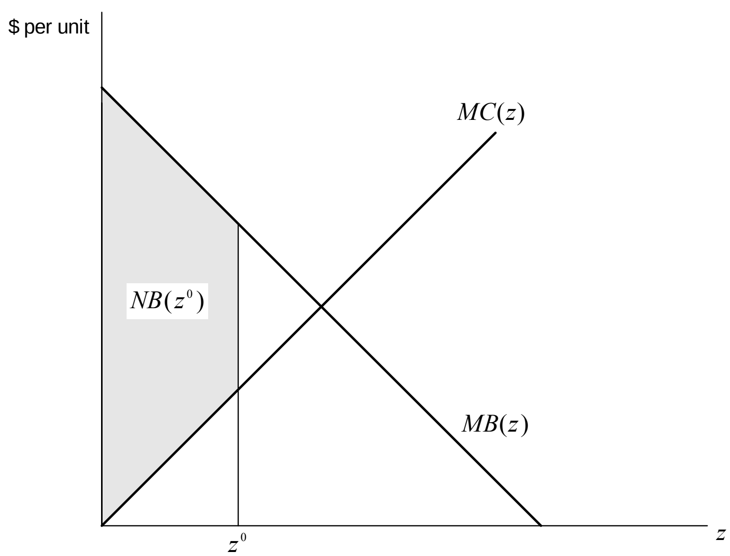


FIGURE A-41

154

Optimization

- Now consider the following experiment:
  - Suppose the household increases its production by a small amount, Δz.
  - This yields an increase in cost and an increase in benefit, equal to ΔC and ΔB respectively; see Figures A-42 and A-43.

155

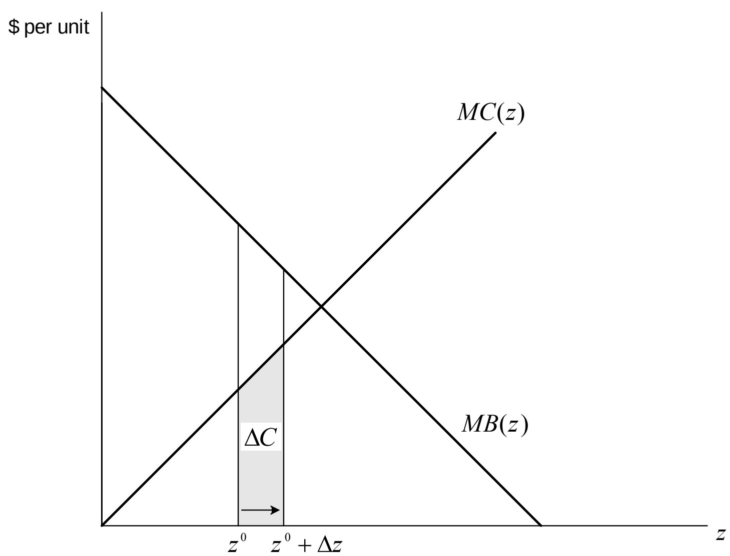


FIGURE A-42

156

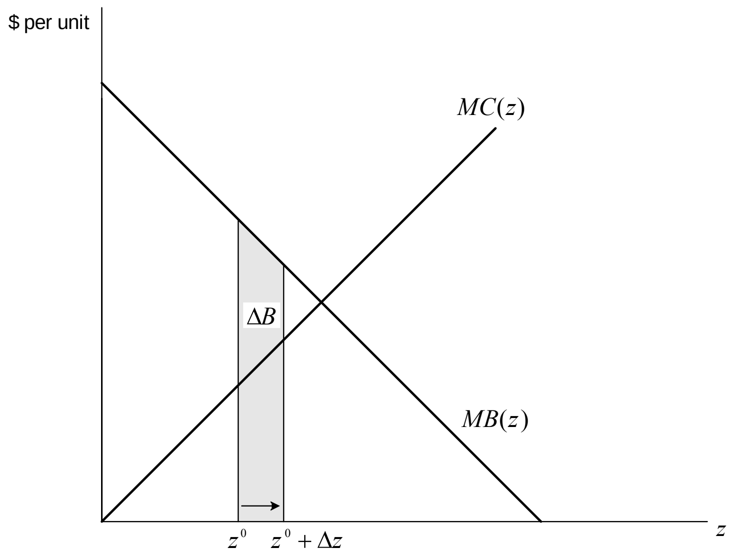


FIGURE A-43

157

Optimization

- Starting at z^0, it is clear from the figures that a small increase in z adds more to benefit than it adds to cost.
- Thus, net benefit rises by ΔNB; see Figure A-44.

158

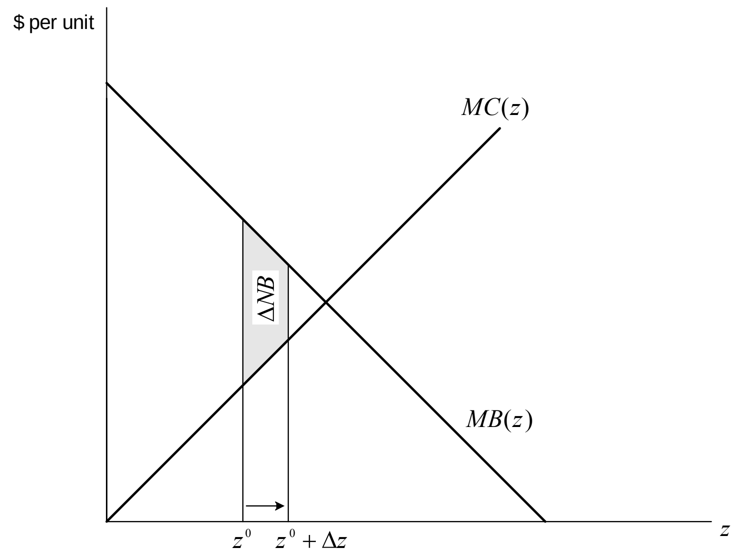


FIGURE A-44

159

Optimization

- The logic is as follows:
  - At z^0, $MB(z^0) > MC(z^0)$
  - This means that a small increase in z leads to an increase in total benefit that exceeds the increase in total cost.
  - It follows that the small increase in z must cause net benefit to rise.
  - Thus, net benefit was <u>not</u> maximized at z^0

160

Optimization

- This same logic holds for <u>any</u> level of z at which $MB(z) > MC(z)$.
- It follows that net benefit cannot be maximized at any level of z at which $MB(z) > MC(z)$.

161

Optimization

- Now consider the opposite scenario, where the current level of electricity generation is at a level where $MB(z) < MC(z)$; see Figure A-45.

162

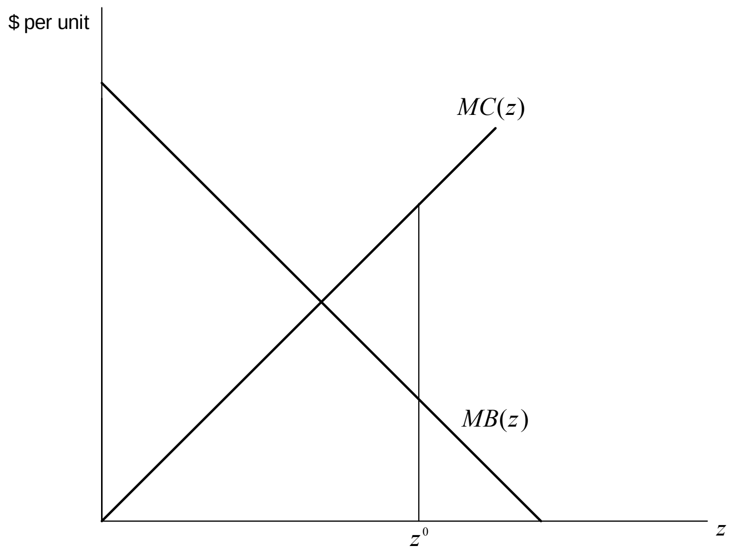


FIGURE A-45

163

Optimization

- Consider the following experiment:
  - Suppose the household <u>reduces</u> its production by a small amount, Δz.
  - This yields a <u>reduction</u> in cost and a <u>reduction</u> in benefit, equal to ΔC and ΔB respectively; see Figures A-46 and A-47.

164

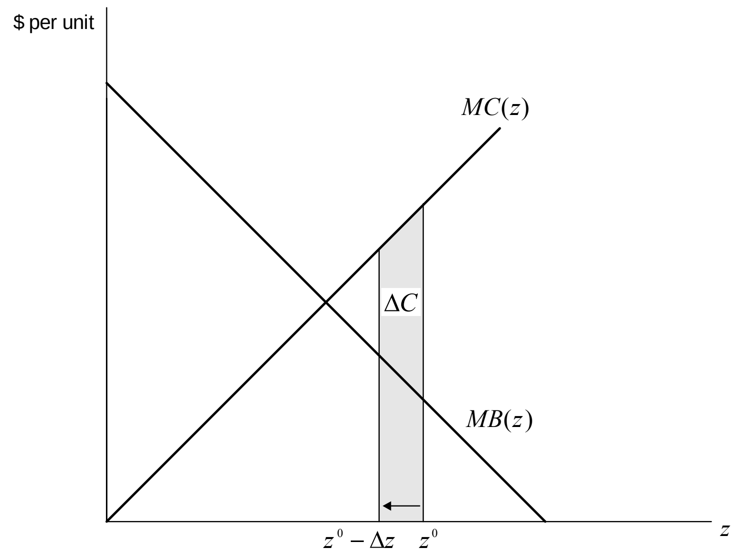


FIGURE A-46 165

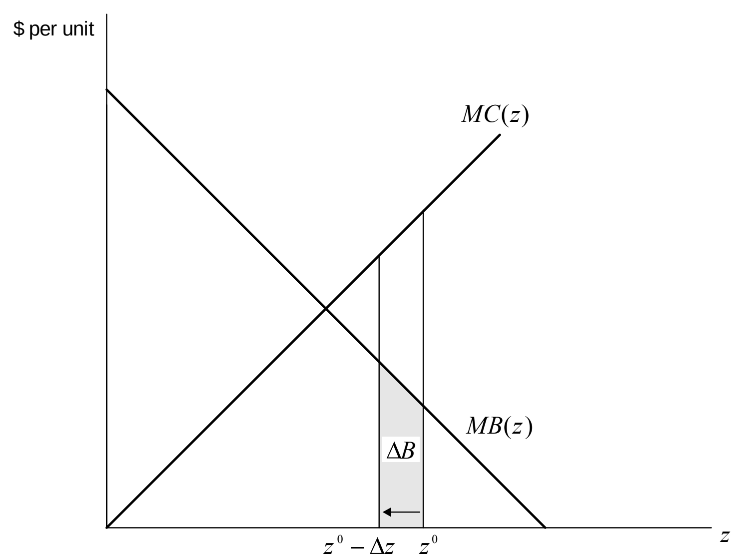


FIGURE A-47 166

Optimization

- Starting at z^0, it is clear from the figures that a small reduction in z reduces cost by more than it reduces benefit.
- Thus, net benefit <u>rises</u> by ΔNB; see Figure A-48.

167

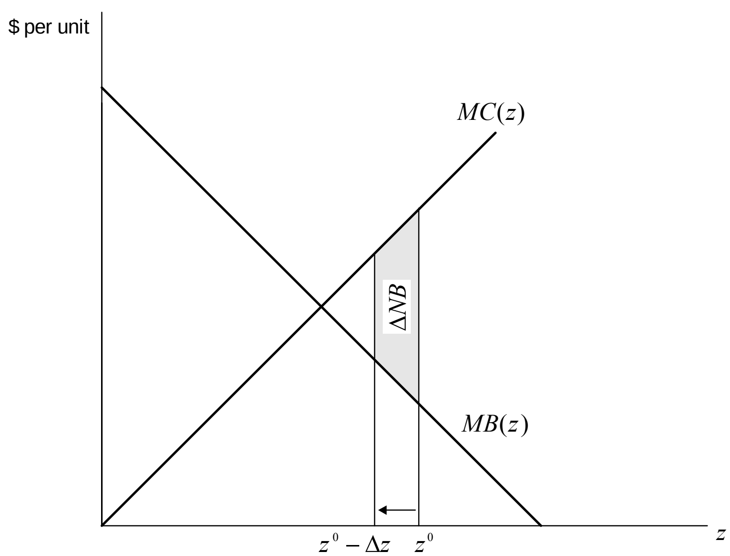


FIGURE A-48

168

Optimization

- The logic is as follows:
  - At z^0, $MC(z^0) > MB(z^0)$
  - This means that a small reduction in z causes a reduction in total cost that exceeds the reduction in total benefit.
  - It follows that the small reduction in z must cause net benefit to rise.
  - Thus, net benefit was <u>not</u> maximized at z^0.

169

Optimization

- This same logic holds for <u>any</u> level of z at which $MC(z) > MB(z)$.
- It follows that net benefit cannot be maximized at any level of z at which $MC(z) > MB(z)$.

170

Optimization

- To summarize the results from our two experiments:
  - Net benefit is not maximized at any z at which $MB(z) > MC(z)$
  - Net benefit is not maximized at any z at which $MC(z) > MB(z)$
- Now consider the knife-edge case where $MC(z) = MB(z)$; see Figure A-49.

171

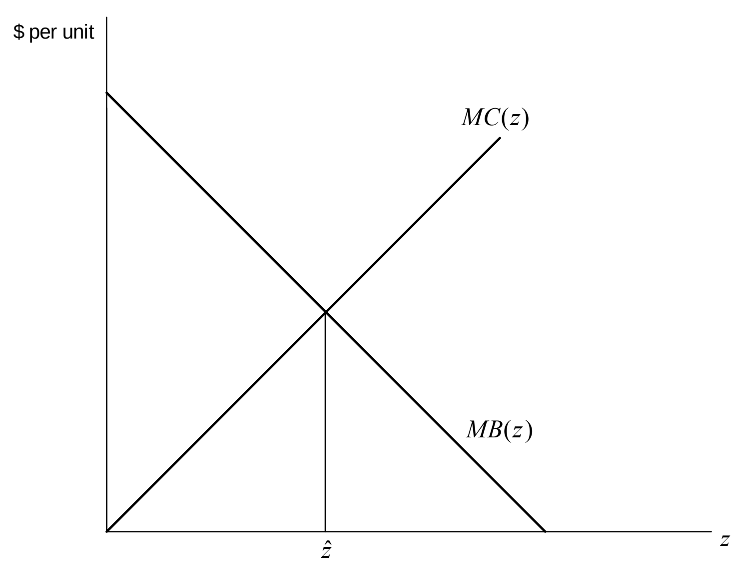


FIGURE A-49

172

Optimization

- In Figure A-49, $MC(z) > MB(z)$ at any $z > z$^.
- This means that a small increase in z causes an increase in total cost that exceeds the increase in total benefit.
- It follows that a small increase in z must cause net benefit to <u>fall</u>; see Figures A-50 and A-51.

173

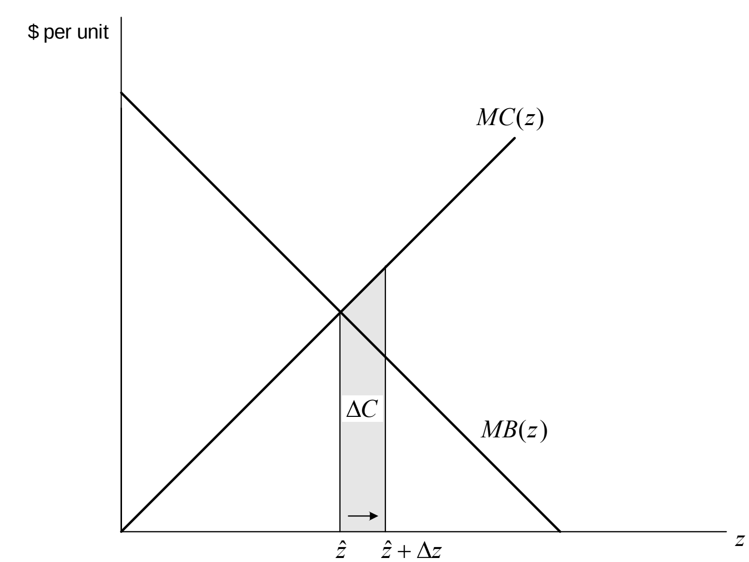


FIGURE A-50

174

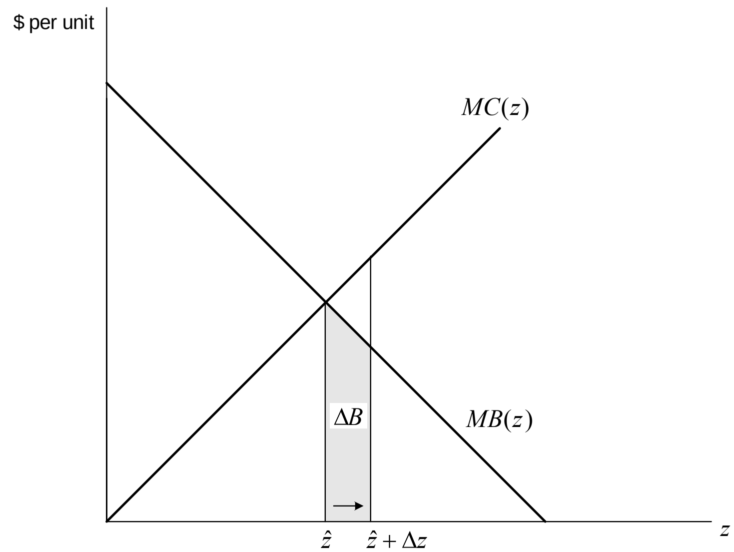


FIGURE A-51

175

Optimization

- Conversely, $MC(z) < MB(z)$ at any $z <$ z^.
- This means that a small <u>reduction</u> in z causes a reduction in total cost that is less than the reduction in total benefit.
- It follows that a small reduction in z must cause net benefit to <u>fall</u>; see Figures A-52 and A-53.

176

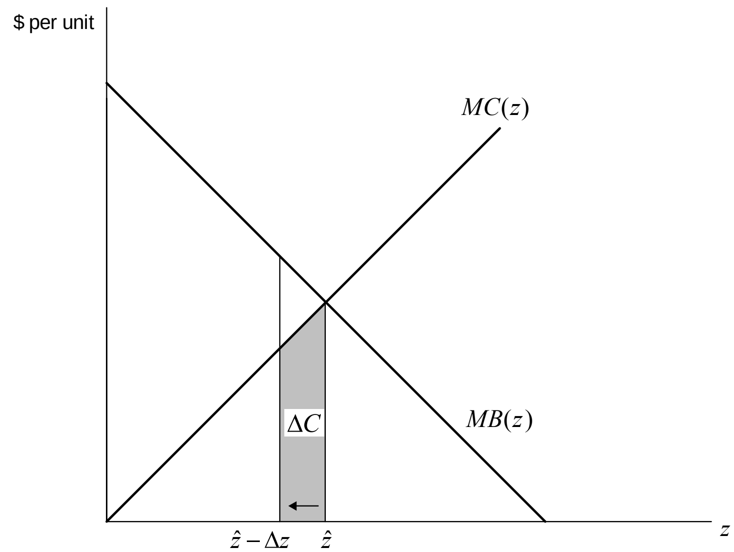


FIGURE A-52 177

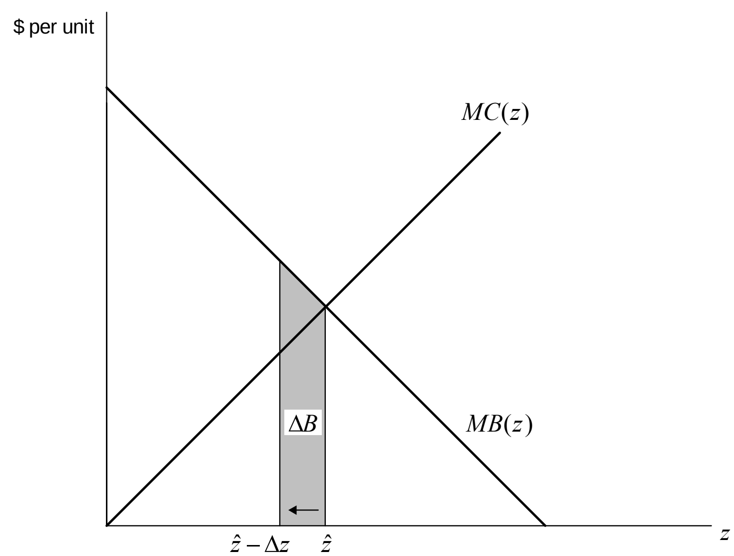


FIGURE A-53 178

Optimization

- We therefore have the following **fundamental result**:
  - Net benefit is maximized at *z*^, where

$$MB(\hat{z}) = MC(\hat{z})$$

179

Optimization

- We can see this same result in a graph of total benefit and total cost.
- Recall that $MB(z)$ measures the slope of a tangent to $B(z)$, and that $MC(z)$ measures the slope of a tangent to $C(z)$.
- These two slopes are equal at *z*^; see Figure A-54.

180

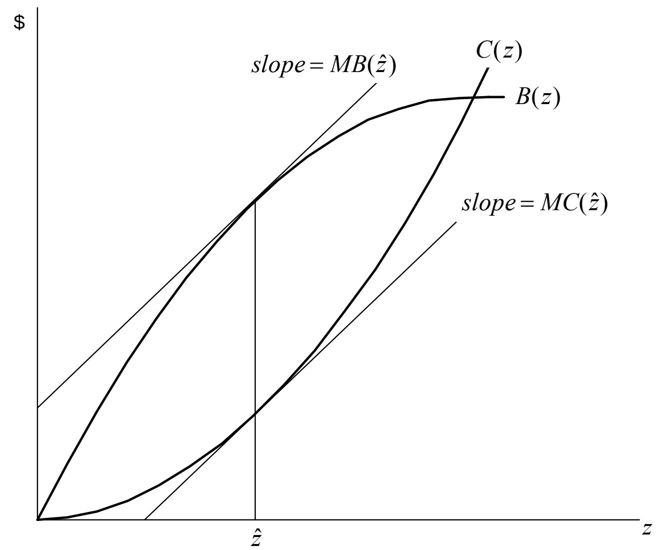


FIGURE A-54

181

Optimization

- At values of *z* below *z*^ in Figure A-54, total benefit is rising faster than total cost as *z* rises (the rate of change of total benefit is higher than the rate of change of total cost).
- Thus, net benefit rises as *z* rises.

182

Optimization

- Conversely, at values of z above z^ in Figure A-54, total cost is falling faster than total benefit as z falls (the rate of change of total cost is higher than the rate of change of total benefit).
- Thus, net benefit rises as z falls.
- It follows that net benefit is maximized at z^.

183

*** Optimization ***

- **If you know calculus**, start with the net benefit function:

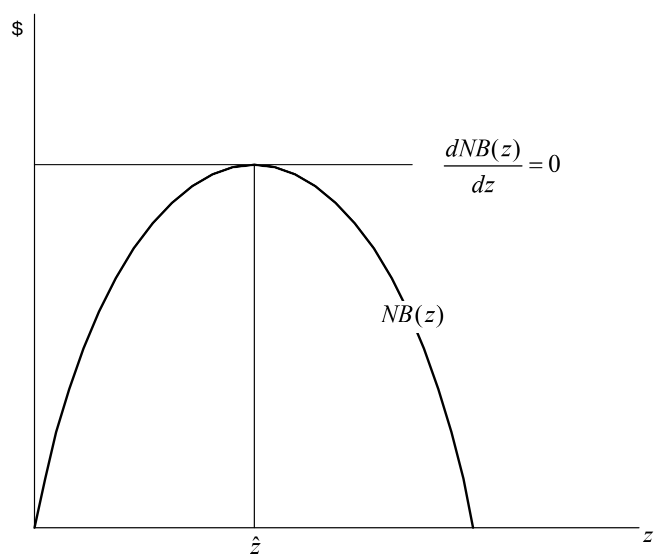
$$NB(z) = B(z) - C(z)$$

184

*** Optimization ***

- To maximize the function, differentiate with respect to z and set the derivative equal to zero.
- This identifies a turning point in $NB(z)$.

185



* FIGURE A-55 *

186

*** Optimization ***

- Taking the derivative yields

$$\frac{dNB(z)}{dz} = \frac{dB(z)}{dz} - \frac{dC(z)}{dz}$$

- Setting this derivative equal to zero yields

$$\frac{dB(z)}{dz} = \frac{dC(z)}{dz}$$

187

*** Optimization ***

- That is,

$$MB(z) = MC(z)$$

- A sufficient (second-order) condition for a maximum is that $NB(z)$ is strictly concave:

$$\frac{d^2 NB(z)}{dz^2} < 0$$

188

*** Optimization ***

- Note that

$$\frac{d^2 NB(z)}{dz^2} = \frac{d^2 B(z)}{dz^2} - \frac{d^2 C(z)}{dz^2}$$

$$= \frac{dMB(z)}{dz} - \frac{dMC(z)}{dz}$$

189

*** Optimization ***

- Thus, sufficient conditions for

$$\frac{d^2 NB(z)}{dz^2} < 0$$

are

$$\frac{dMB(z)}{dz} < 0 \qquad \text{and} \qquad \frac{dMC(z)}{dz} > 0$$

190

*** Optimization ***

- That is, $MB(z)$ is downward-sloping, and $MC(z)$ is upward-sloping, as per our graphs.
- Note that these are sufficient but not necessary conditions; we require only that $NB(z)$ is strictly concave.

191

Optimization

- Let us illustrate this characterization of the optimum in the context of our electricity example (see Figure A-56).

192

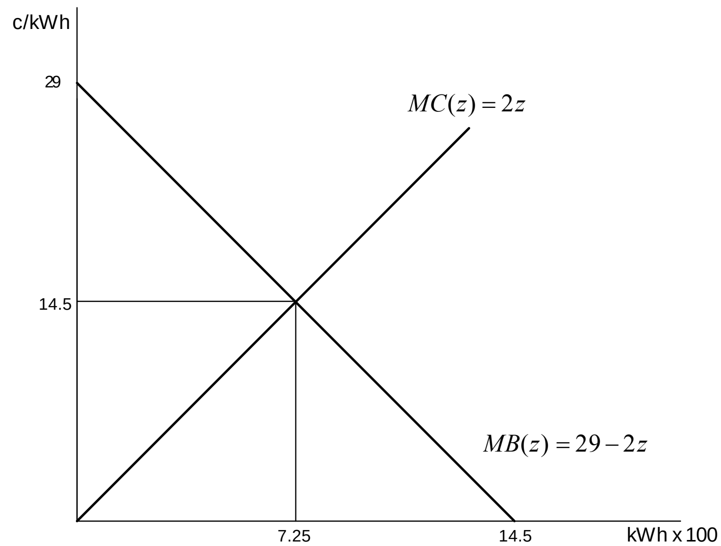


FIGURE A-56

193

Optimization

- Finding the solution in Figure A-56 – the intersection of the two graph – requires a knowledge of linear functions.
- We will encounter functions of this type throughout the course, so it is important to have a good understanding of them.
- Let us review the essentials.

194

PART 2: WORKING WITH LINEAR FUNCTIONS

2.1 SLOPES AND INTERCEPTS

Working with Linear Functions

- The general form of a linear function* is

$$f(x) = ax + b$$

where a is the slope and b is the vertical intercept when $f(x)$ is plotted against x.
- See Figure A-57 for the case where $a>0$.

* Technically, this is an "affine" function if b is not zero

197

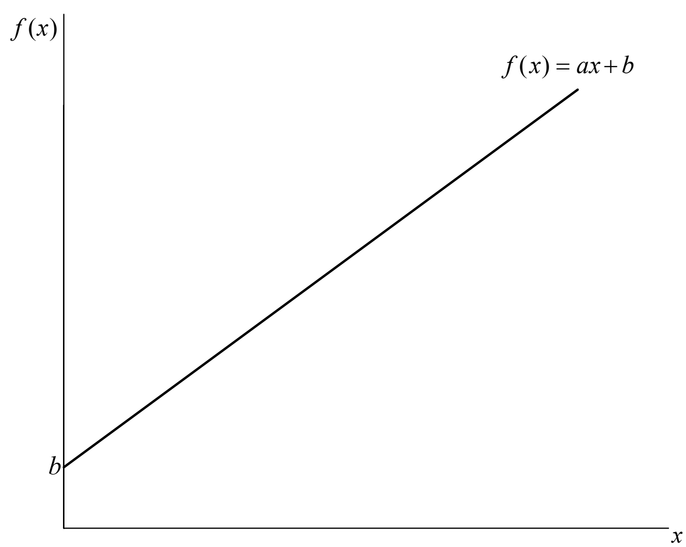


FIGURE A-57

198

Working with Linear Functions

- The slope of the function has the following graphical interpretation:

$$\textbf{slope} = \frac{\textbf{vertical rise}}{\textbf{horizontal run}}$$

199

Working with Linear Functions

- Thus, for any change in x (denoted Δx) we can calculate the change in the value of the function (denoted $\Delta f(x)$)

$$\Delta f(x) = a\Delta x$$

- See Figure A-58.

200

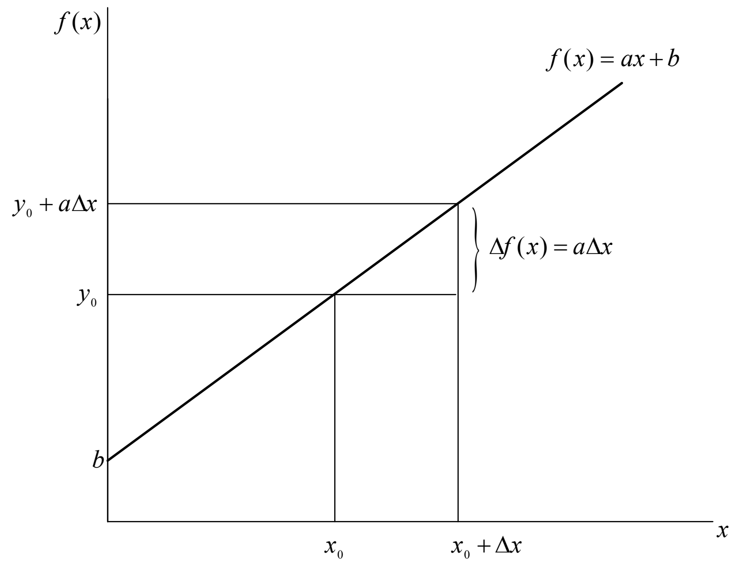


FIGURE A-58 201

2.2 CALCULATING AREAS

Calculating Areas

- We can also easily calculate areas under a linear function because for linear functions, these areas comprise combinations of rectangles and triangles.
- Consider the shaded area in Figure A-59.

203

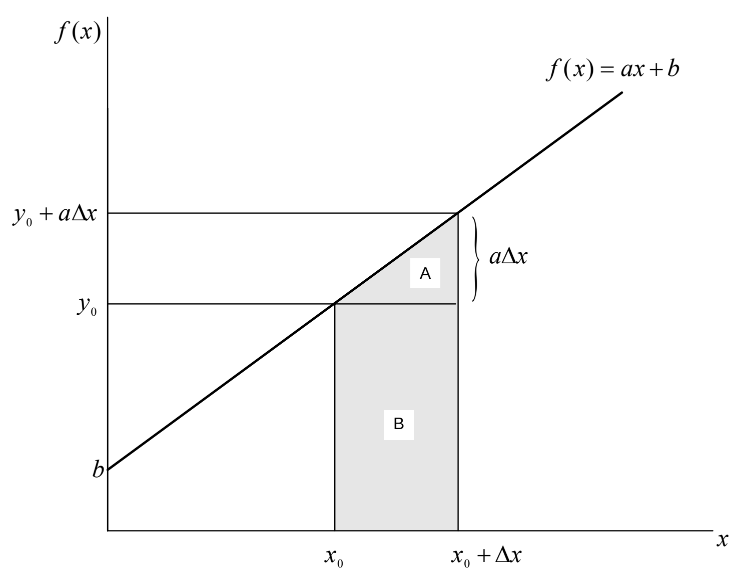


FIGURE A-59

204

Calculating Areas

- This area is calculated as the sum of areas A and B, where

$$A = \frac{\Delta x(a\Delta x)}{2} = \frac{a(\Delta x)^2}{2}$$

$$B = \Delta x(y_0)$$

205

Calculating Areas

- Thus, the area in Figure A-59 is

$$\int_{x_0}^{x_0+\Delta x} f(x)dx = \frac{a(\Delta x)^2}{2} + \Delta x(y_0)$$

206

Calculating Areas

- Now consider the case where $a<0$; see Figure A-60.

207

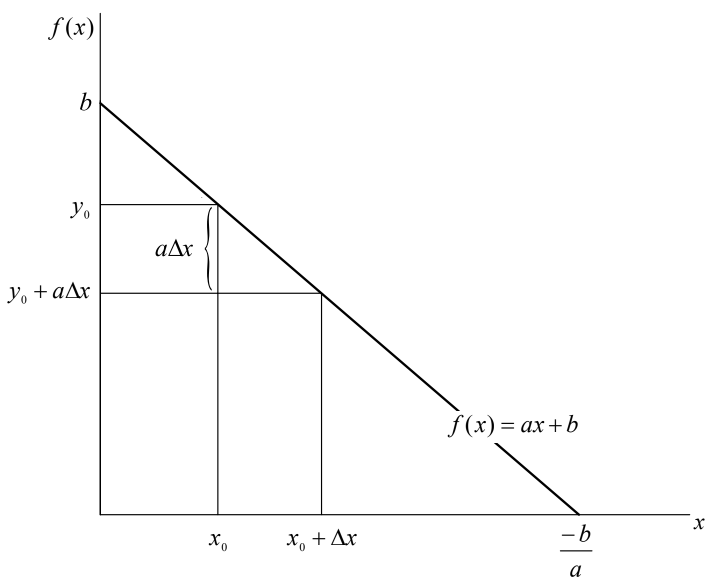


FIGURE A-60

208

Calculating Areas

- Consider the area in Figure A-61.

209

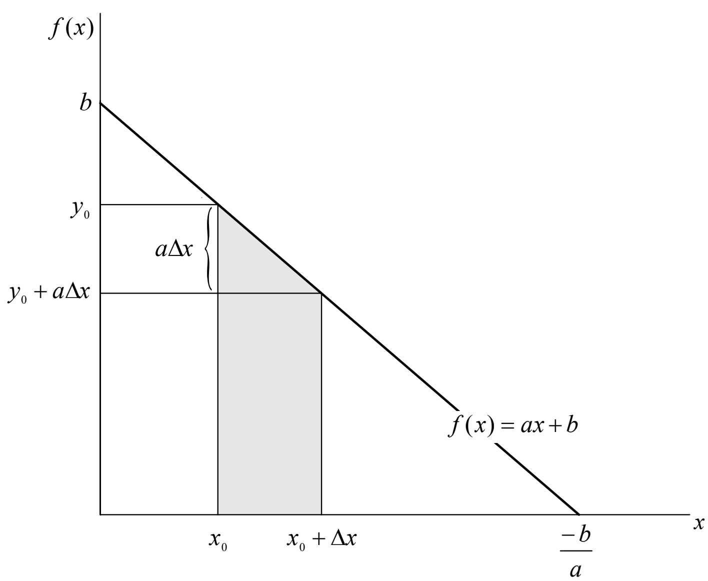


FIGURE A-61

210

Calculating Areas

- This is calculated using the <u>same formula</u> as for the *a*>0 case:

$$\int_{x_0}^{x_0+\Delta x} f(x)dx = \frac{a(\Delta x)^2}{2} + \Delta x(y_0)$$

where the first RHS term is negative when *a*<0; see Figures A-62 and A-63.

211

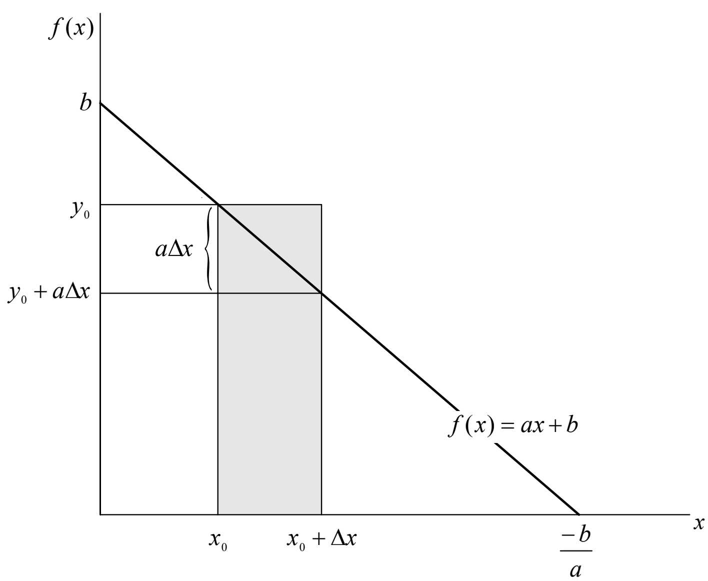


FIGURE A-62

212

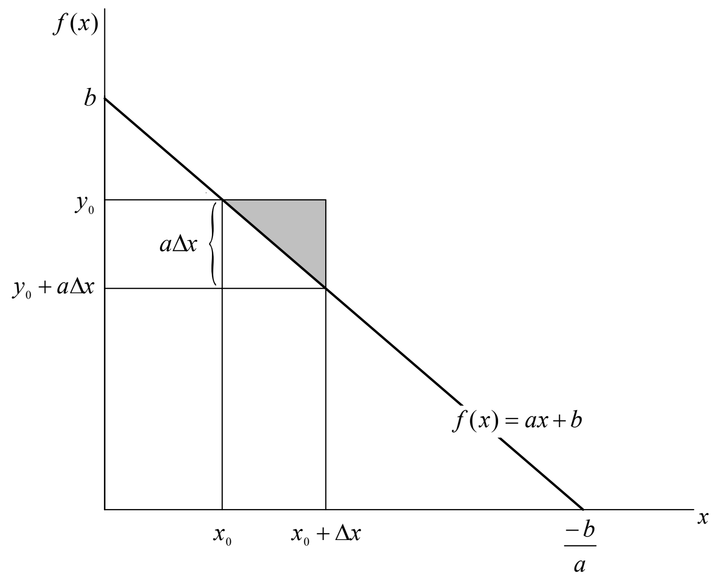


FIGURE A-63

213

Calculating Areas

- Similarly, the area to the left of a graph can be calculated as the sum of two areas; see Figure A-64.

214

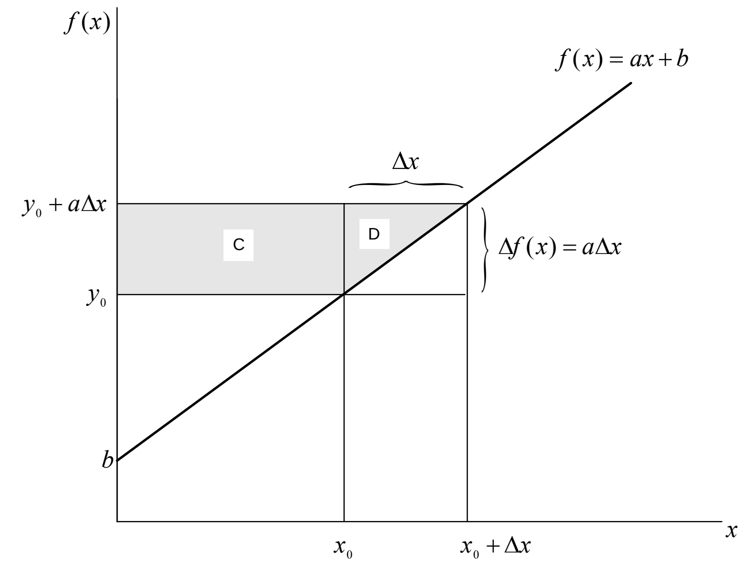


FIGURE A-64

215

Calculating Areas

- The area in Figure A-64 is calculated as the sum of areas C and D, where

$$C = x_0(a\Delta x)$$

$$D = \frac{\Delta x(a\Delta x)}{2} = \frac{a(\Delta x)^2}{2}$$

216

2.3 SOLVING PAIRS OF EQUATIONS

Solving Pairs of Equations

- Now suppose we have a pair of linear functions, and we wish to solve for where they intersect.

218

Solving Pairs of Equations

- Suppose our two functions are

$$f(x)_1 = a_1 x + b_1$$

$$f(x)_2 = a_2 x + b_2$$

where $a_1>0$ and $a_2<0$; see Figure A-65.

219

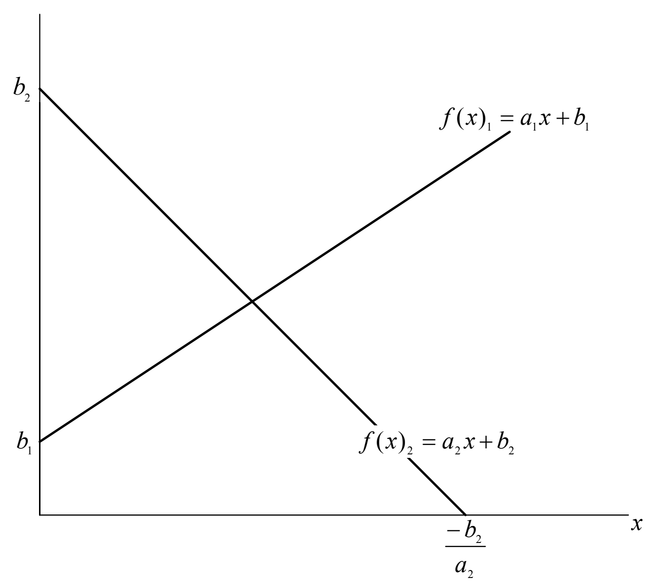


FIGURE A-65

220

Solving Pairs of Equations

- The intersection of these two functions occurs where

$$a_1 x + b_1 = a_2 x + b_2$$

- Collecting terms, we then have

$$a_1 x - a_2 x = b_2 - b_1$$

221

Solving Pairs of Equations

- Solving for x yields

$$\hat{x} = \frac{b_2 - b_1}{a_1 - a_2}$$

222

Solving Pairs of Equations

- Evaluating $f(x)_1$ at this value yields

$$f(\hat{x})_1 = a_1\left(\frac{b_2 - b_1}{a_1 - a_2}\right) + b_1$$

$$= a_1\left(\frac{b_2 - b_1}{a_1 - a_2}\right) + b_1\left(\frac{a_1 - a_2}{a_1 - a_2}\right) = \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}$$

223

Solving Pairs of Equations

- Evaluating $f(x)_2$ at this value yields

$$f(\hat{x})_2 = a_2\left(\frac{b_2 - b_1}{a_1 - a_2}\right) + b_2$$

$$= a_2\left(\frac{b_2 - b_1}{a_1 - a_2}\right) + b_2\left(\frac{a_1 - a_2}{a_1 - a_2}\right) = \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}$$

224

Solving Pairs of Equations

- Thus,

$$f(\hat{x})_1 = f(\hat{x}_2)$$

which confirms that our solution for x is correct; see Figure A-66.

225

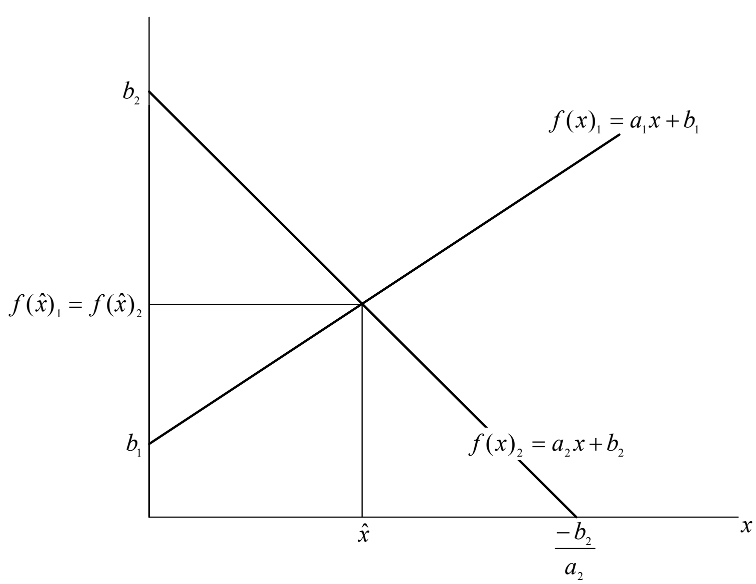


FIGURE A-66

226

Solving Pairs of Equations

- As a numerical example, recall Figure A-56, where our two functions are

$$MC(z) = 2z$$

$$MB(z) = 29 - 2z$$

- In this example, a_1=2, a_2= –2, b_1=0 and b_2=29.

227

Solving Pairs of Equations

- The intersection occurs where

$$29 - 2\hat{z} = 2\hat{z}$$

- Thus, the optimal solution for our household (measured in kWh x 100) is

$$\hat{z} = 7.25$$

228

Solving Pairs of Equations

- The total benefit at this optimum is the shaded area in Figure A-67, denoted *B*(*z*^).

229

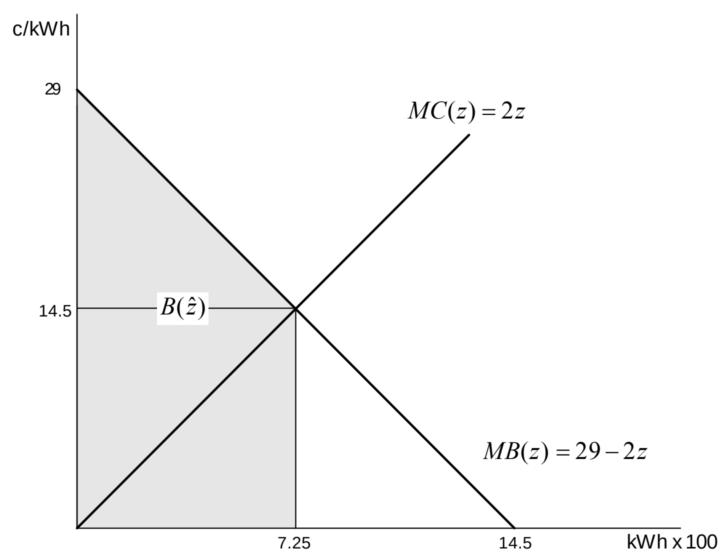


FIGURE A-67

230

Solving Pairs of Equations

- The total cost at the optimum is the shaded area in Figure A-68, denoted $C(z^\wedge)$.

231

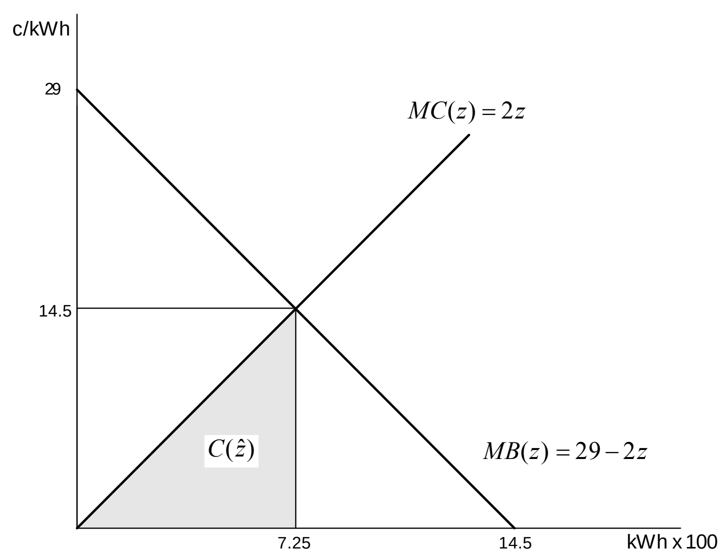


FIGURE A-68

232

Solving Pairs of Equations

- The net benefit at the optimum is the difference between these two areas, denoted *NB*(z^) in Figure A-69.

233

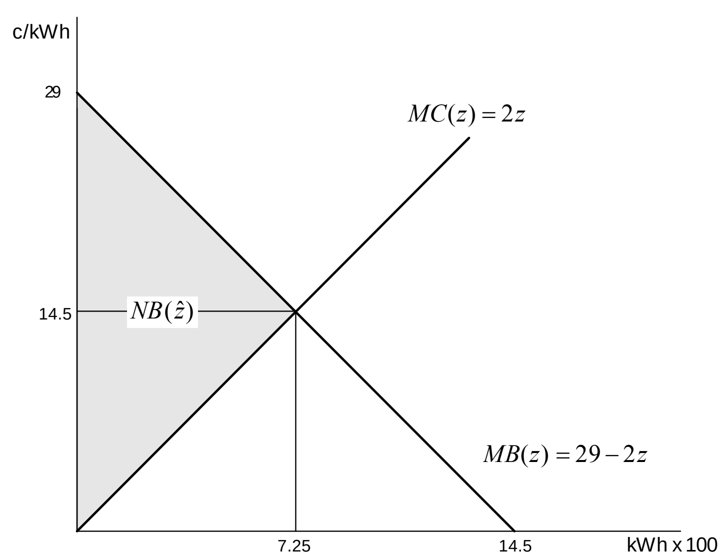


FIGURE A-69

234

Solving Pairs of Equations

- Calculating these areas yields

$$B(z^\wedge) = \$157.6875$$
$$C(z^\wedge) = \$52.5625$$
$$NB(z^\wedge) = \$105.125$$

235

2.4 VERTICAL SUMMATION

Vertical Summation

- We will often find it necessary to work with the sum of two functions or the difference between them.
- For example, let us construct a function $g(x)$ such that

$$g(x) = f(x)_1 + f(x)_2$$

237

Vertical Summation

- Suppose

$$f(x)_1 = a_1 x + b_1$$

$$f(x)_2 = a_2 x + b_2$$

238

Vertical Summation

- Then

$$g(x) = (a_1 + a_2)x + (b_1 + b_2)$$

- Thus, $g(x)$ has slope a_1+a_2, and vertical intercept b_1+b_2.

239

Vertical Summation

- Graphically, $g(x)$ is the <u>vertical sum</u> of $f(x)_1$ and $f(x)_2$, as illustrated in Figure A-70 for a case where $a_1>0$ and $a_2<0$.

240

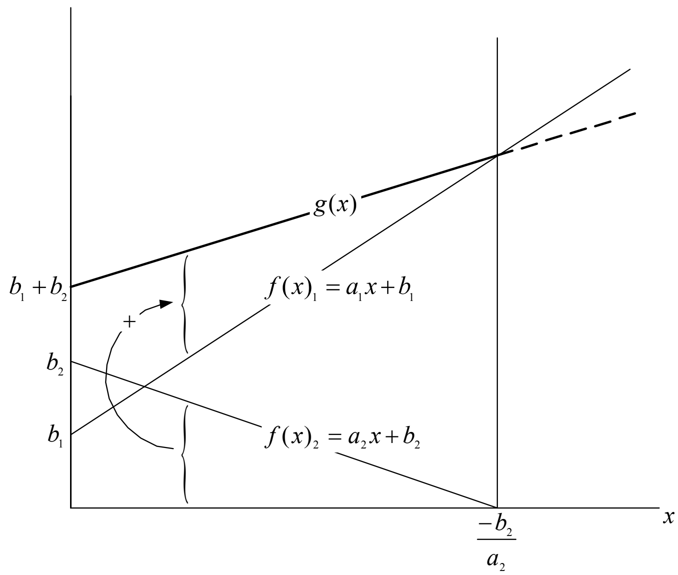


FIGURE A-70

241

Vertical Summation

- Similarly, we can construct a function $h(x)$ as the difference between our two functions:

$$h(x) = f(x)_1 - f(x)_2$$
$$= (a_1 - a_2)x + (b_1 - b_2)$$

- See Figure A-71.

242

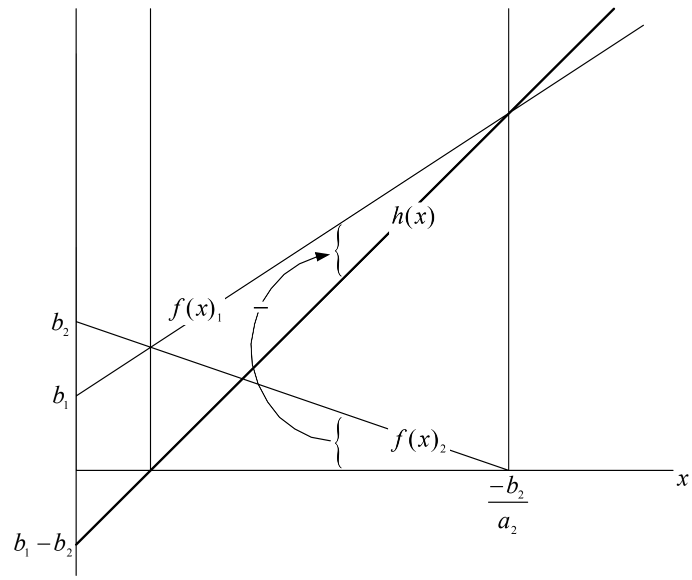


FIGURE A-71

243

Vertical Summation

- Consider a numerical example.
- Suppose

$$f(x)_1 = 4x + 2$$

$$f(x)_2 = -2x + 14$$

244

Vertical Summation

- Then

$$f(x)_1 + f(x)_2 = (4x+2)+(-2x+14) = 2x+16$$

$$f(x)_2 - f(x)_1 = (-2x+14)-(4x+2) = 12-6x$$

- See Figures A-72 and A-73 respectively.

245

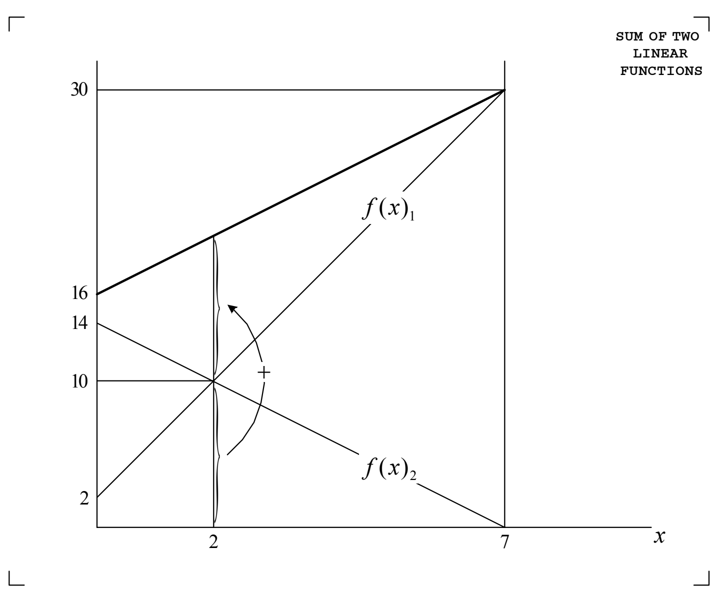


FIGURE A-72

246

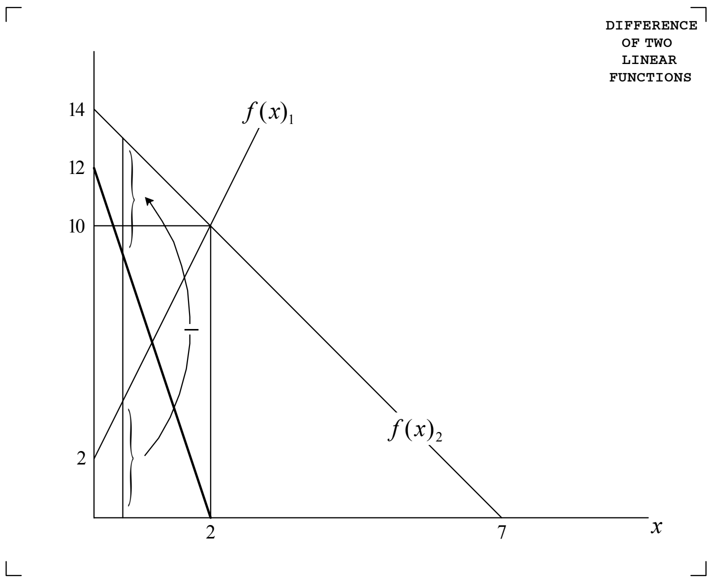


FIGURE A-73 247

2.5 HORIZONTAL SUMMATION

Horizontal Summation

- Now suppose we have two functions in <u>two variables</u> x_1 and x_2:

$$f(x_1)_1 = a_1 x_1 + b_1$$

$$f(x_2)_2 = a_2 x_2 + b_2$$

where x_1 and x_2 are measured in the same units.

249

Horizontal Summation

- For example, these two functions might represent marginal willingness-to-pay (WTP) of two individuals, where $f(x_i)_i$ is the WTP of person i for a marginal unit of consumption, as a function of the amount x_i that she currently consumes.
- See Figure A-74.

250

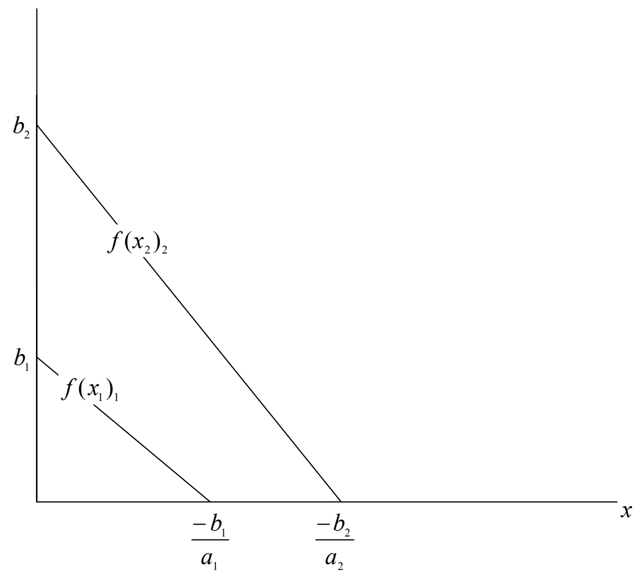


FIGURE A-74

251

Horizontal Summation

- In the context of the demand example, we would measure $f(x_i)_i$ in dollars per unit, and use p_i to denote the dependent variable:

$$p_i = a_i x_i + b_i$$

- We interpret this as an **inverse demand curve** for person i, relating demand x_i to the price p_i she faces in the market.

252

Horizontal Summation

- Now suppose both individuals face the same price in the market, such that

$$p_1 = p_2 = p$$

- We can then construct an **aggregate demand curve** as the horizontal summation of the two individual demand curves.

253

Horizontal Summation

- In what sense is this a horizontal summation?
- At any given value of p, we add the horizontal distances measured off the two individual demand curves to construct a new aggregate consumption variable $X = x_1 + x_2$ as a function of p; see Figure A-75.

254

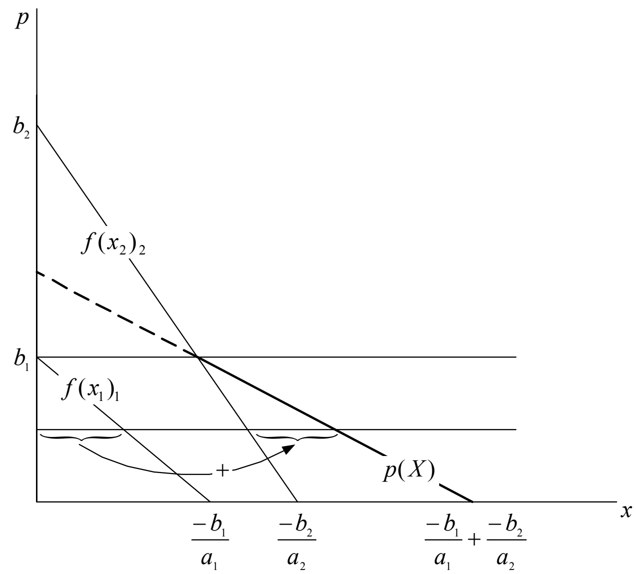


FIGURE A-75

255

Horizontal Summation

- We will return to the question of why $p(X)$ is a drawn as a dashed line above b_1 in Figure A-75 (see s.270).
- Right now, our interest is in how we find the formula for the aggregate function, $p(X)$.

256

Horizontal Summation

- We solve the problem by first recognizing that the horizontal summation is actually the <u>vertical</u> summation of the <u>inverse</u> functions, where *x* is plotted on the vertical axis and *p* is plotted on the horizontal axis; see Figure A-76 for this inversion of Figure A-75.

257

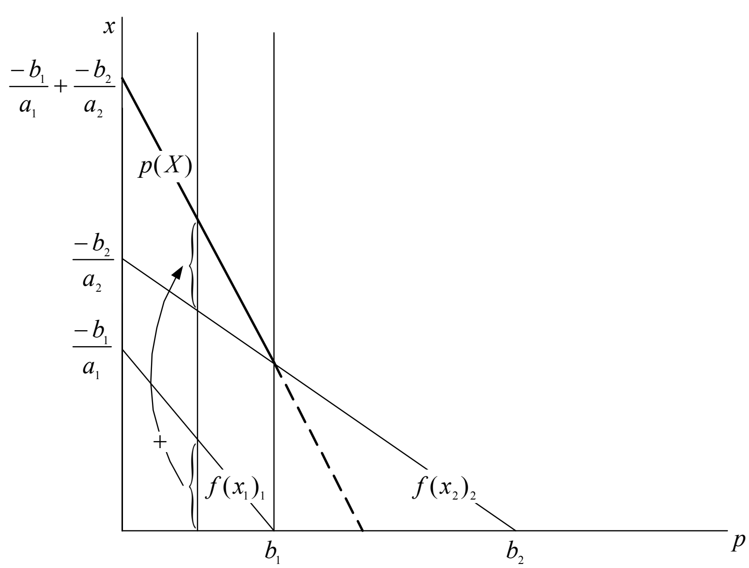


FIGURE A-76

258

Horizontal Summation

- The second step is to construct the inverse demand functions for each person analytically.

259

Horizontal Summation

- For person *i*:

$$p = a_i x_i + b_i \quad \rightarrow \quad p - b_i = a_i x_i$$

$$\rightarrow \quad x_i = \frac{p - b_i}{a_i} \quad \rightarrow \quad x_i(p) = \alpha_i p + \beta_i$$

where $\alpha_i = \dfrac{1}{a_i}$ and $\beta_i = \dfrac{-b_i}{a_i}$

260

Horizontal Summation

- Now take the <u>vertical</u> summation of $x_1(p)$ and $x_2(p)$ to find $X(p)$:

$$X(p) = x_1(p) + x_2(p)$$
$$= (\alpha_1 p + \beta_1) + (\alpha_2 p + \beta_2)$$
$$= (\alpha_1 + \alpha_2)p + (\beta_1 + \beta_2)$$

261

Horizontal Summation

- Now take the inverse of this function to express p as a function X:

$$X - (\beta_1 + \beta_2) = (\alpha_1 + \alpha_2)p$$

$$\rightarrow \quad p = \frac{X}{\alpha_1 + \alpha_2} - \frac{\beta_1 + \beta_2}{\alpha_1 + \alpha_2}$$

262

Horizontal Summation

- Thus, we have

$$p(X) = AX + B$$

where $A = \dfrac{1}{\alpha_1 + \alpha_2}$ and $B = -\dfrac{\beta_1 + \beta_2}{\alpha_1 + \alpha_2}$

263

Horizontal Summation

- Substituting for α_i and β_i from s.257 we have

$$A = \frac{1}{\alpha_1 + \alpha_2} = \frac{1}{\left(\frac{1}{a_1} + \frac{1}{a_2}\right)} = \frac{a_1 a_2}{a_1 + a_2}$$

and $$B = -\frac{a_1 b_2 + a_2 b_1}{a_1 + a_2}$$

264

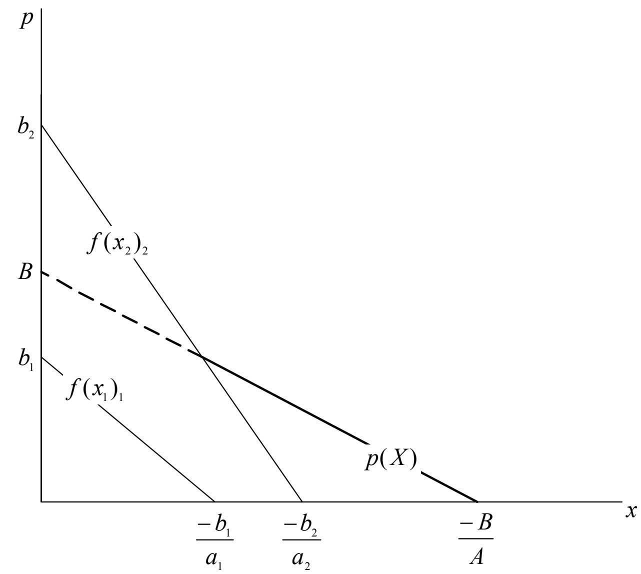


FIGURE A-77

265

Horizontal Summation

- Consider an example. Suppose

$$p(x_1)_1 = 12 - 6x_1$$

is the marginal WTP of person 1 for bread, where x_1 is her consumption of bread.

266

Horizontal Summation

- Similarly, suppose

$$p(x_2)_2 = 15 - 3x_2$$

is the marginal WTP of person 2 for bread, where x_2 is his consumption of bread.
- See Figure A-78.

267

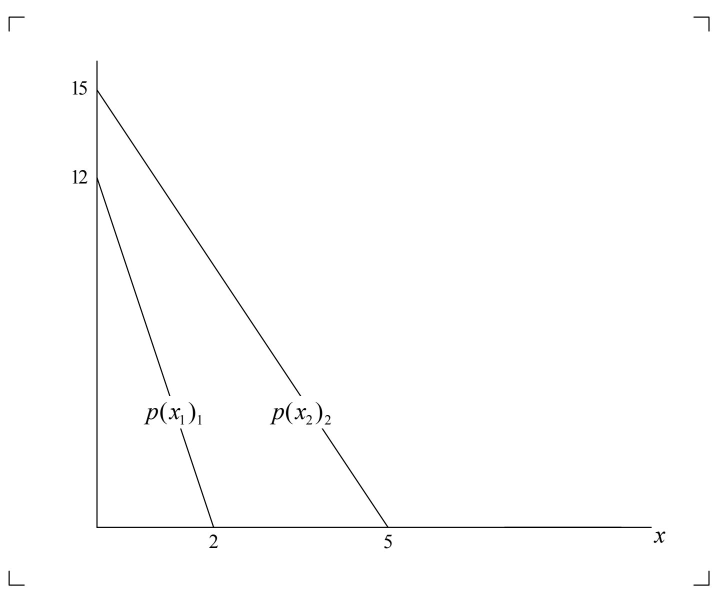


FIGURE A-78

268

Horizontal Summation

- These WTP functions are interpreted as <u>inverse demand functions</u>.
- Now let us construct the <u>demand functions</u> for each person (the inverse of the inverse demand functions).
- These demand functions expressed quantity consumed as a function of price, p.

269

Horizontal Summation

- For person 1:

$$p = 12 - 6x_1 \;\rightarrow\; x_1 = \frac{12 - p}{6} \;\rightarrow\; x_1(p) = 2 - \frac{p}{6}$$

- For person 2:

$$p = 15 - 3x_2 \;\rightarrow\; x_2 = \frac{15 - p}{3} \;\rightarrow\; x_2(p) = 5 - \frac{p}{3}$$

270

Horizontal Summation

- We can now construct aggregate demand:

$$X(p) = x_1(p) + x_2(p) = 7 - \frac{p}{2}$$

- This tells us the total quantity consumed at any given price.

271

Horizontal Summation

- Now construct the inverse of this aggregate demand function:

$$X = 7 - \frac{p}{2} \rightarrow \frac{p}{2} = 7 - X \rightarrow p(X) = 14 - 2X$$

- This the inverse aggregate demand function; see Figure A-79.

272

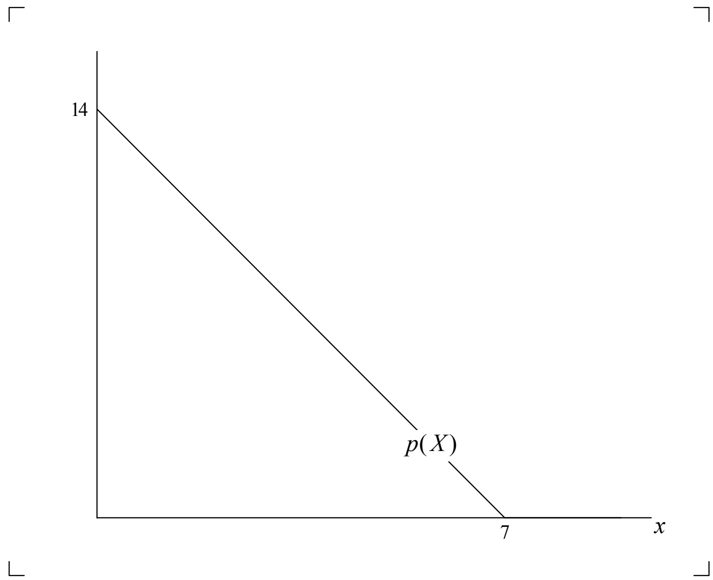


FIGURE A-79

273

Horizontal Summation

- This inverse aggregate demand is the horizontal summation of two individual inverse demand functions; see Figure A-80.

274

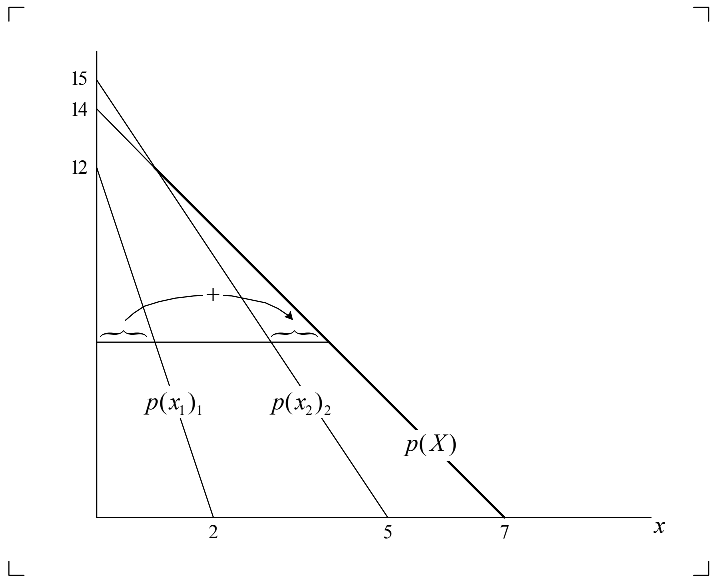


FIGURE A-80

275

Horizontal Summation

- Why is $p(X)$ dashed above p=12?
- Above p=12, person 1 consumes zero but our simple mathematical calculations treat her consumption at p>12 as negative.
- Thus, above p=12 the true inverse aggregate demand coincides with the inverse demand of person 2; see Figure A-81.

276

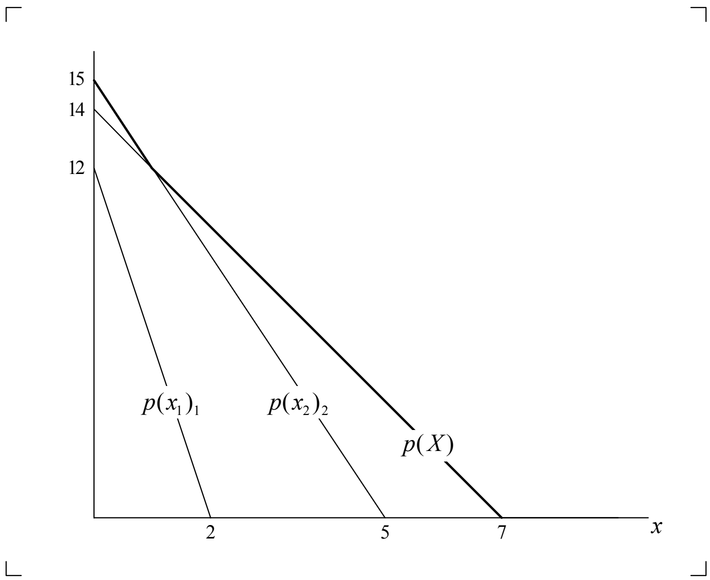


FIGURE A-81 277

PART 3: REVIEW QUESTIONS

Questions 1 – 15 relate to the following data. The marginal benefit and marginal cost of some activity *x* are given by

$$MB(x) = 154 - 5x \qquad \text{and} \qquad MC(x) = 10 + 7x$$

respectively. Figure 1 provides a graph of each function.

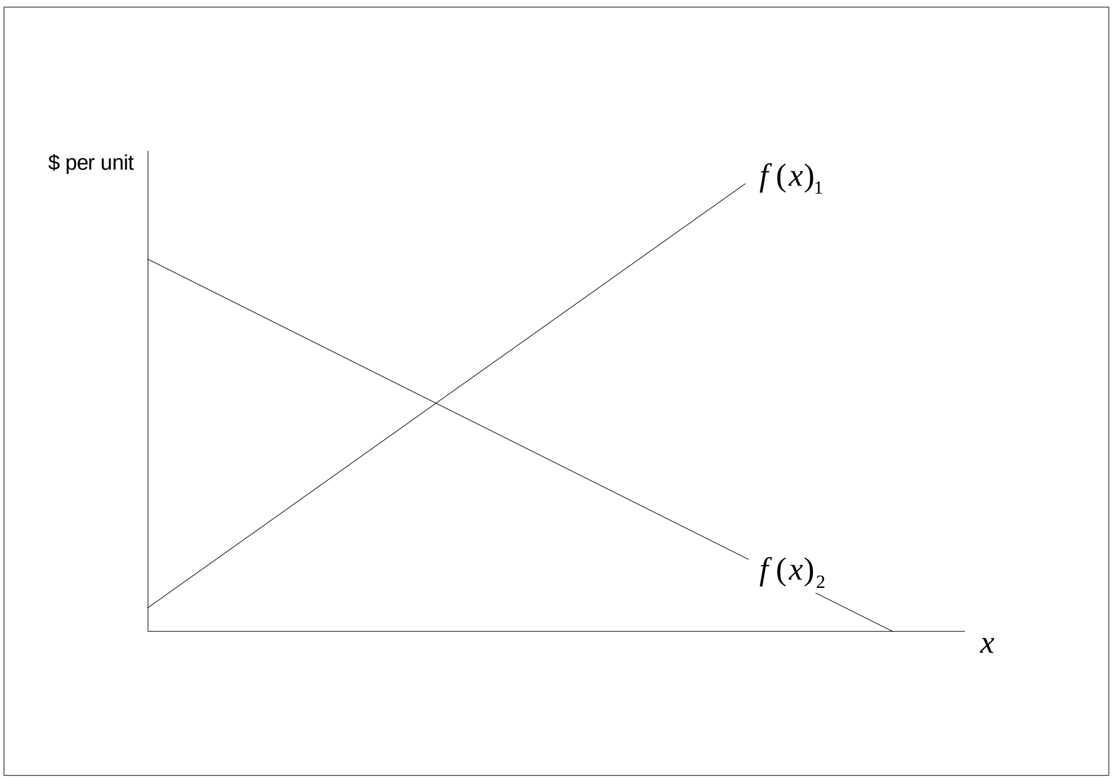


FIGURE 1

1. The function labeled $f(x)_1$ is the best representation of $MB(x)$.
A. True
B. False

2. The vertical intercept for $MC(x)$ is
A. 10/7
B. 7
C. 10
D. None of the above

3. The horizontal intercept for $MB(x)$ is
A. 154
B. 5/154
C. -5
D. None of the above

4. The optimal solution for x (where marginal benefit and marginal cost are equated) is
A. 10
B. 12
C. 15
D. 20

5. Marginal benefit at the optimum is
A. 149
B. 10
C. 94
D. 12

6. Marginal cost at the optimum is
A. 3
B. 94
C. 27
D. 10

7. Total benefit at the optimum is
A. 1488
B. 624
C. 864
D. None of the above

8. Total cost at the optimum is
A. 1488
B. 624
C. 864
D. None of the above

9. Total net benefit at the optimum is
A. 1488
B. 624
C. 864
D. None of the above

10. The rate of change of the benefit function at the optimum is
A. 149
B. 10
C. 94
D. 12

11. Let $H(x) = MB(x) - MC(x)$ denote the vertical difference between $MB(x)$ and $MC(x)$. Then
A. $H(x) = 164 + 2x$
B. $H(x) = 144 - 12x$
C. $H(x) = 144 - 2x$
D. $H(x) = 164 - 12x$

12. The horizontal intercept of $H(x)$ is
A. 0
B. 72
C. 41/3
D. 12

13. Compare your answer to Q.12 with your answer to Q.4. Is this relationship a coincidence?
A. Yes
B. No

14. The area $\int_0^{12} H(x)dx$ is equal to
A. 1488
B. 624
C. 864
D. None of the above

15. Compare your answer to Q.14 with your answer to Q.9. Is this relationship a coincidence?
A. Yes
B. No

Questions 16 – 37 relate to the following data. The marginal benefit and marginal cost of some activity *x* are given by

$$MB(x) = 36 - 3x \qquad \text{and} \qquad MC(x) = 9x$$

respectively. Figure 2 provides a graph of each function.

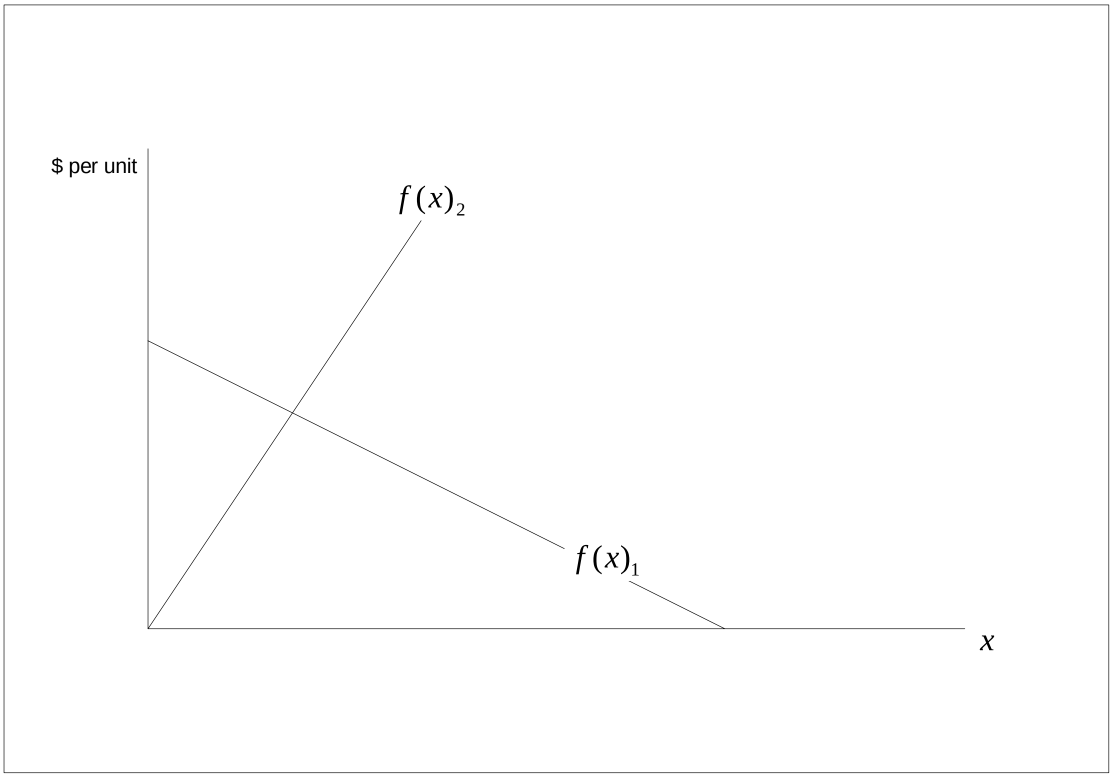


FIGURE 2

16. The function labeled $f(x)_1$ is the best representation of $MB(x)$.
A. True
B. False

17. The vertical intercept for $MC(x)$ is
A. 0
B. 36
C. 12
D. None of the above

18. The horizontal intercept for $MB(x)$ is
A. 12
B. 36
C. 3
D. 9

19. The optimal solution for x (where marginal benefit and marginal cost are equated) is
A. 0
B. 1
C. 2
D. 3

20. Marginal benefit at the optimum is
A. 36
B. 27
C. 9
D. 0

21. Marginal cost at the optimum is
A. 3
B. 9
C. 12
D. None of the above

22. Total benefit at the optimum is
A. 54
B. 81/2
C. 189/2
D. None of the above

23. Total cost at the optimum is
A. 54
B. 81/2
C. 189/2
D. None of the above

24. Total net benefit at the optimum is
A. 54
B. 81/2
C. 189/2
D. None of the above

25. The rate of change of the cost function at the optimum is
A. 9
B. 3
C. 12
D. 27

26. Let $W(x) = MB(x) - MC(x)$ denote the vertical difference between $MB(x)$ and $MC(x)$. Then
A. $W(x) = -36 - 6x$
B. $W(x) = 36 + 3x$
C. $W(x) = -36 + 12x$
D. $W(x) = 36 - 12x$

27. The horizontal intercept of $H(x)$ is
A. 0
B. 9
C. 3
D. 12

28. Compare your answer to Q27 with your answer to Q19. Is this relationship a coincidence?
A. Yes
B. No

29. The area $\int_0^3 W(x)dx$ is equal to
A. 54
B. 81/2
C. 189/2
D. None of the above

30. Compare your answer to Q29 with your answer to Q24. Is this relationship a coincidence?
A. Yes
B. No

Now suppose we introduce an additional marginal cost function, given by

$$MD(x) = 6x$$

31. Let $S(x) = MC(x) + MD(x)$ denote the <u>revised</u> cost, equal to the vertical sum of $MC(x)$ and $MD(x)$. Then

A. $S(x) = 3x$
B. $S(x) = 54x$
C. $S(x) = 6 + 9x$
D. $S(x) = 15x$

32. The <u>revised</u> optimum is the solution to $MB(x) = S(x)$. It is

A. 0
B. 1
C. 2
D. 3

33. Total benefit at the revised optimum is

A. 30
B. 36
C. 66
D. 96

34. Revised total cost at the revised optimum is

A. 30
B. 36
C. 66
D. 96

35. The two shaded areas in Figure 3 are equal.

A. True
B. False

36. Revised total net benefit at the revised optimum is

A. 30
B. 36
C. 66
D. 96

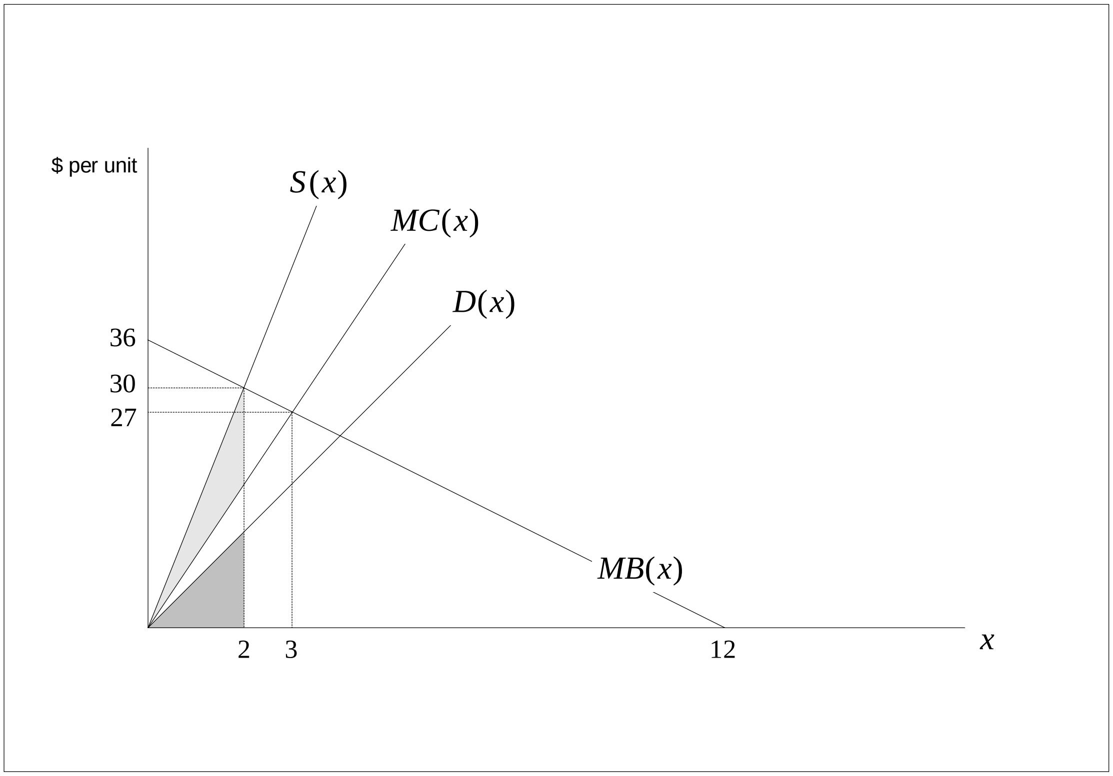


FIGURE 3

37. (The absolute value of) the area $\int_2^3 MB(x)dx - \int_2^3 S(x)dx$ is equal to

A. 3
B. 6
C. 9
D. 6

Questions 38 – 40 relate to the following data. Two firms each use an input *x* to produce output. The inverse demands for this input are

$$w(x_1)_1 = 40 - 2x_1 \qquad \text{and} \qquad w(x_2)_2 = 50 - 8x_2$$

respectively. Let *w* denote the price paid for this input. Let *X* denote $x_1 + x_2$.

38. The demand for *x* by firm 1 is

A. $x_1(w) = 90 - 10w$

B. $x_1(w) = \frac{1}{40} - \frac{w}{2}$

C. $x_1(w) = 20 - \frac{w}{2}$

D. $x_1(w) = 40 - 2w$

39. The demand for *x* by firm 2 is

A. $x_2(w) = \frac{25}{4} - \frac{w}{8}$

B. $x_2(w) = 50 - 8w$

C. $x_2(w) = \frac{1}{50} - \frac{w}{8}$

D. $x_2(w) = \frac{50}{8} - w$

40. The inverse aggregate demand for *x* is

A. $X(w) = \frac{125}{4} - \frac{5w}{8}$

B. $w(X) = 42 - \frac{8X}{5}$

C. $w(X) = 90 - 10X$

D. $X(w) = 9 - \frac{w}{10}$

ANSWER KEY

1. **B**. The $MB(x)$ has negative slope but $f(x)_1$ has positive slope.

2. **C**. See Figure RA-1.

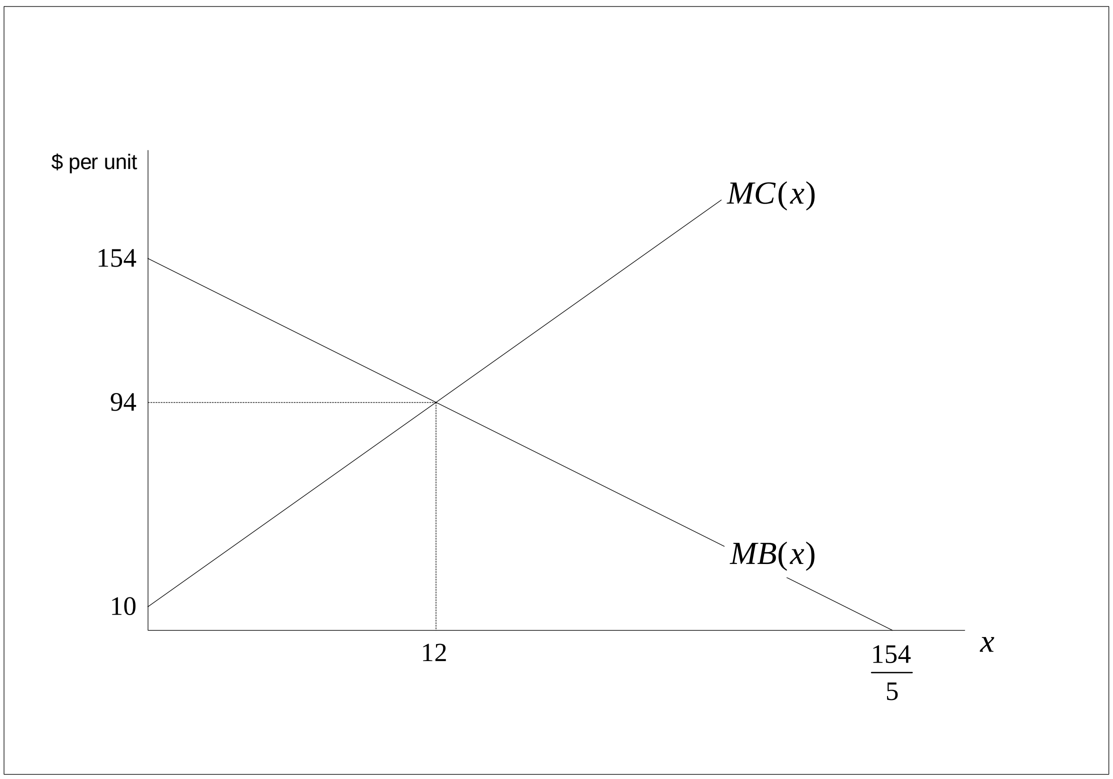


FIGURE RA-1

3. **D**. The horizontal intercept is calculated by setting $MB(x) = 0$ and solving for *x*. This yields 154/5. See Figure RA-1.

4. **B**. The optimal value solves $MB(x) = MC(x)$. This yields $\hat{x} = 12$. See Figure RA-1.

5. **C**. $MB(\hat{x}) = 154 - 5(12) = 94$. See Figure RA-1.

6. **B**. $MC(\hat{x}) = 10 + 7(12) = 94$. See Figure RA-1.

7. **A**. This is the shaded area in Figure RA-2. It comprises

$$A = \frac{60(12)}{2} \quad \text{plus} \quad B = 94(12)$$

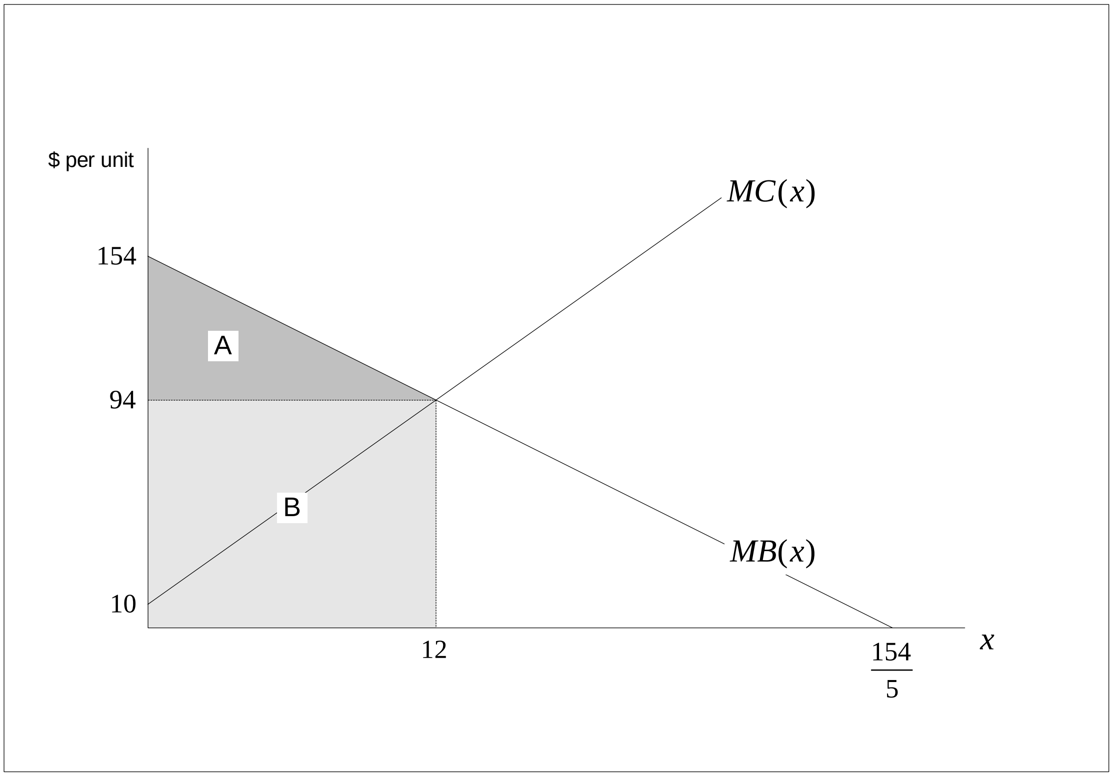


FIGURE RA-2

8. **B**. This is the shaded area in Figure RA-3. It comprises

$$A = \frac{84(12)}{2} \quad \text{plus} \quad B = 10(12)$$

9. **C**. This is the shaded area in Figure RA-4. It is equal to difference between the areas in Figure RA-2 and Figure RA-3.

10. **C**. The rate of change of the benefit function is the marginal benefit function. Evaluated at $\hat{x} = 12$, it is equal to 94.

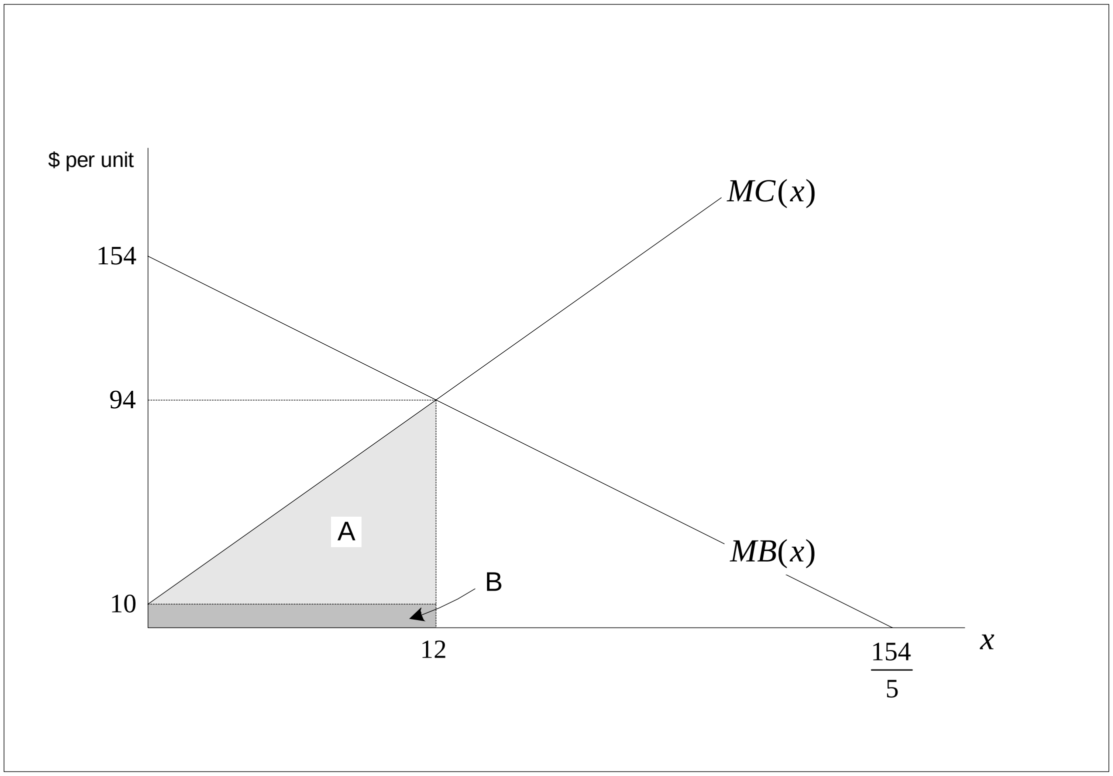


FIGURE RA-3

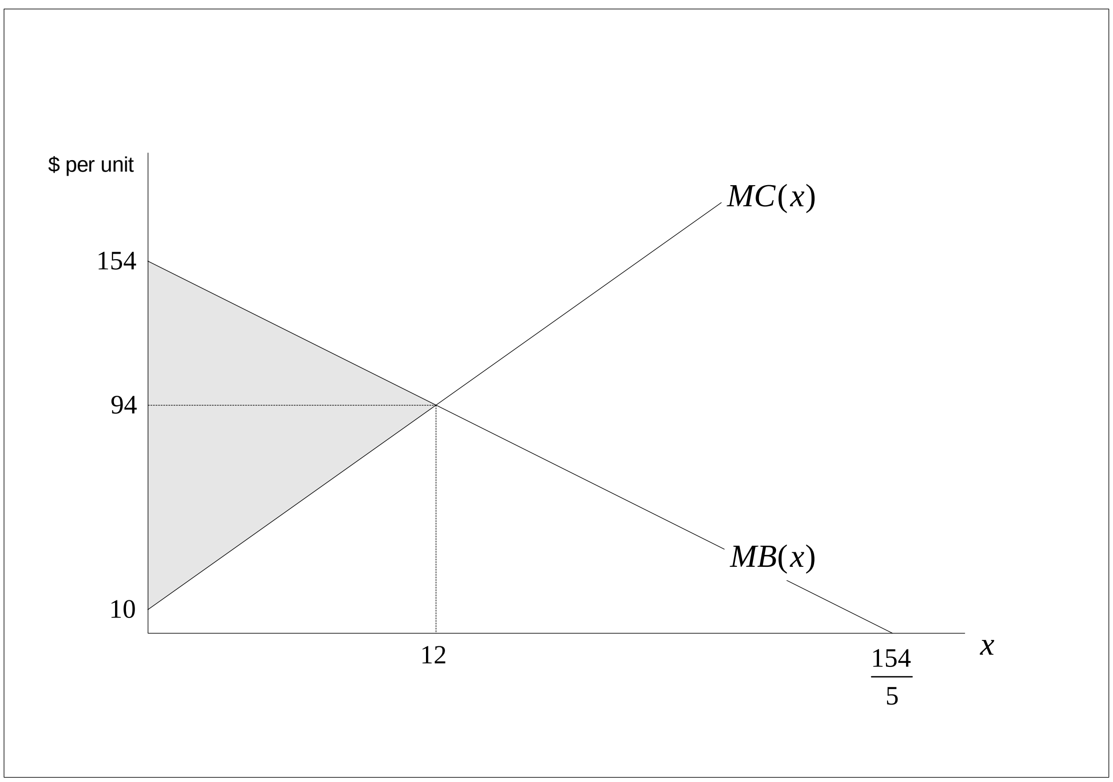


FIGURE RA-4

11. **B**. $H(x) = MB(x) - MC(x) = (154-5x)-(10+7x) = 144-12x$. It is depicted in Figure RA-5.

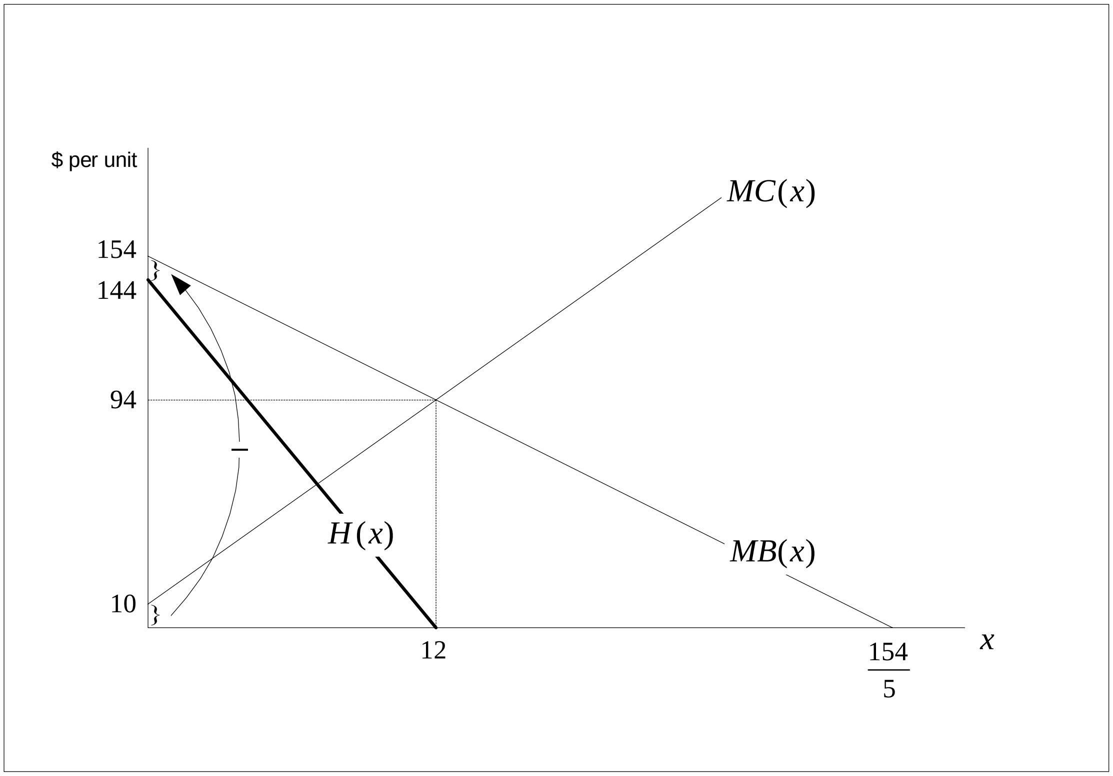


FIGURE RA-5

12. **D**. See Figure RA-5.

13. **B**. The horizontal intercept of $H(x)$ occurs where $H(x) = 0$. Since $H(x) = MB(x) - MC(x)$, it follows that the intercept of $H(x)$ is at $\hat{x} = 12$, where $MB(x) = MC(x)$.

14. **C**. There are two ways to arrive at this answer: the elegant way and the inelegant way. First, the elegant way. $H(x)$ is the difference between two functions, $MB(x) - MC(x)$, so the area under $H(x)$ must be equal to the difference in the areas under $MB(x)$ and $MC(x)$, which we have already calculated (in Q.7 and Q.8 respectively). Putting these words into math:

$$\int_0^{12} H(x)dx = \int_0^{12} [MB(x) - MC(x)]dx = \int_0^{12} MB(x)dx - \int_0^{12} MC(x)dx = 1488 - 624 = 864$$

Now the inelegant way:

$\int_0^{12} H(x)dx$ is the shaded area in Figure RA-6. Its area is $\dfrac{144(12)}{2} = 864$

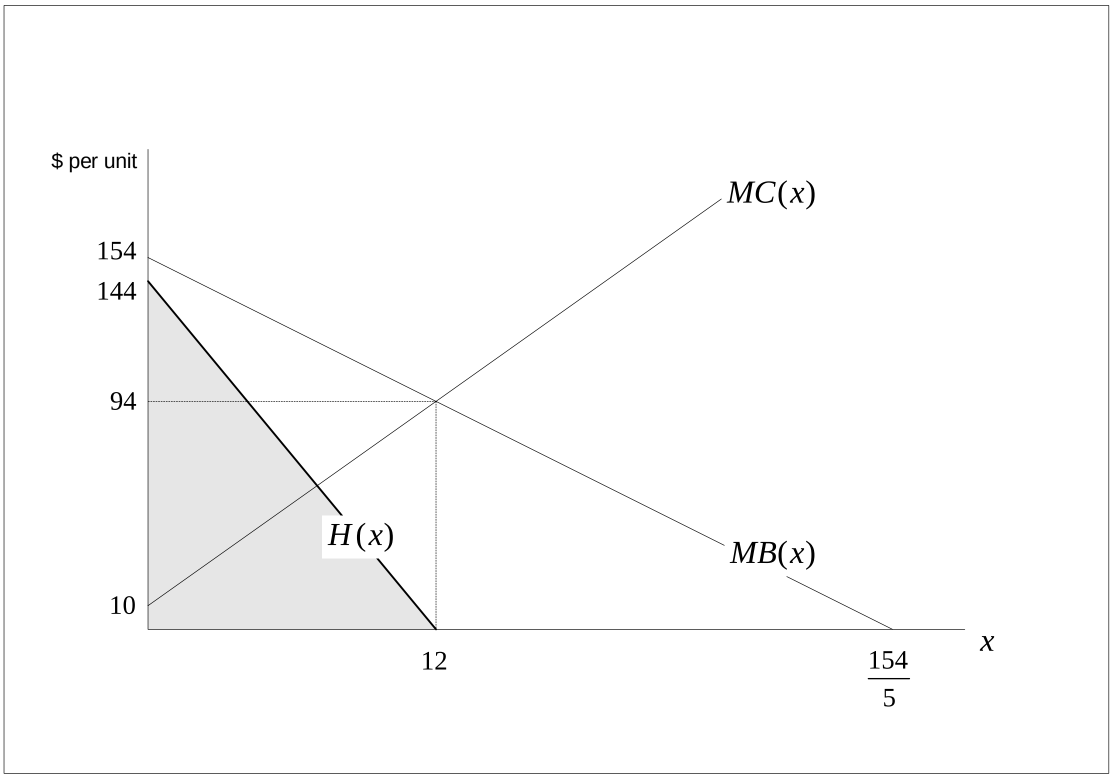


FIGURE RA-6

15. **B**. See the reasoning in the answer to Q.14

16. **A**. It has negative slope.

17. **A**.

18. **A**. Set $36 - 3x = 0$ to yield $x = 12$.

19. **D**. The optimal value solves $MB(x) = MC(x)$. This yields $\hat{x} = 3$. See Figure RA-7.

20. **B**. $MB(\hat{x}) = 36 - 3(3) = 27$. See Figure RA-7.

21. **D**. $MC(\hat{x}) = 9(3) = 27$. See Figure RA-7.

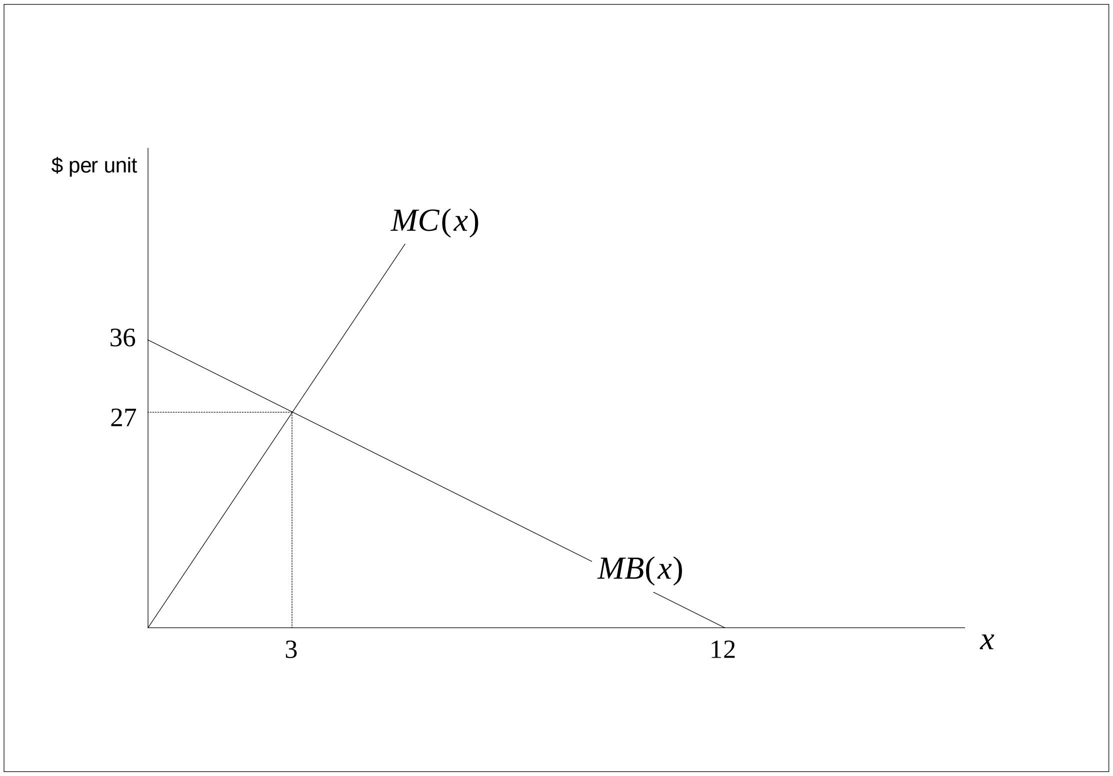


FIGURE RA-7

22. **C**. This is the shaded area in Figure RA-8. It comprises

$$A = \frac{9(3)}{2} \quad \text{plus} \quad B = 27(3)$$

23. **B**. This is the shaded area in Figure RA-9. It is equal to

$$\frac{27(3)}{2}$$

24. **A**. This is the shaded area in Figure RA-10. It is equal to difference between the areas in Figure RA-8 and Figure RA-9.

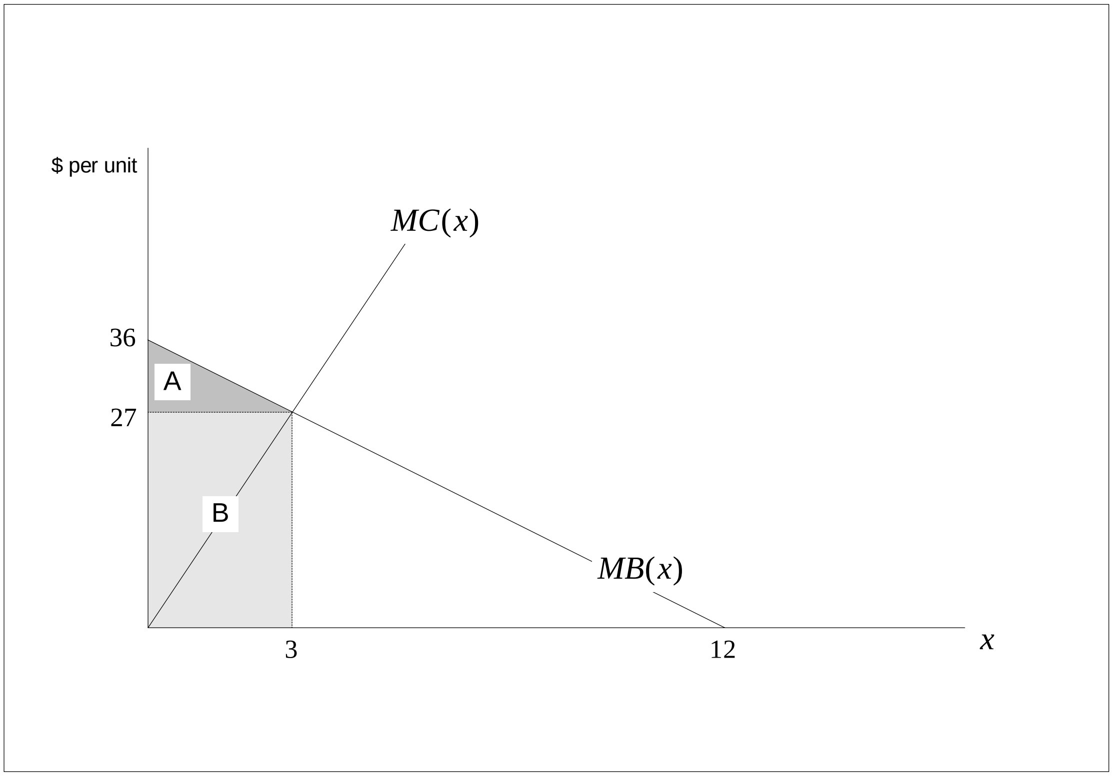


FIGURE RA-8

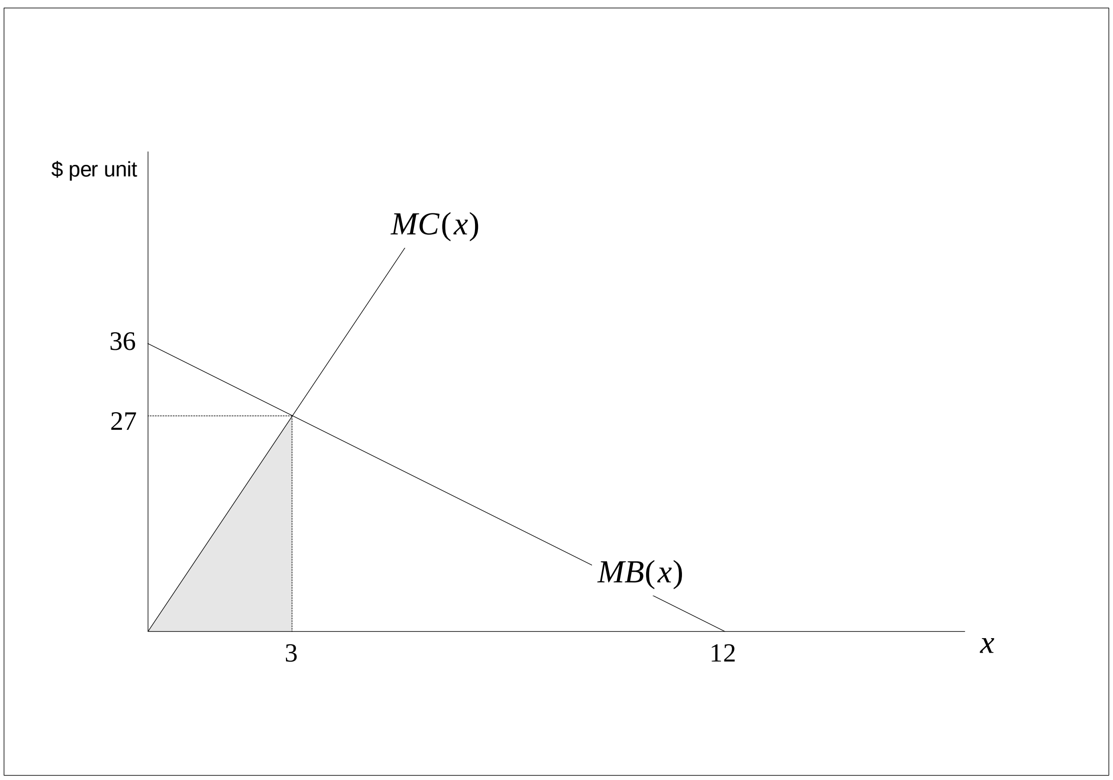


FIGURE RA-9

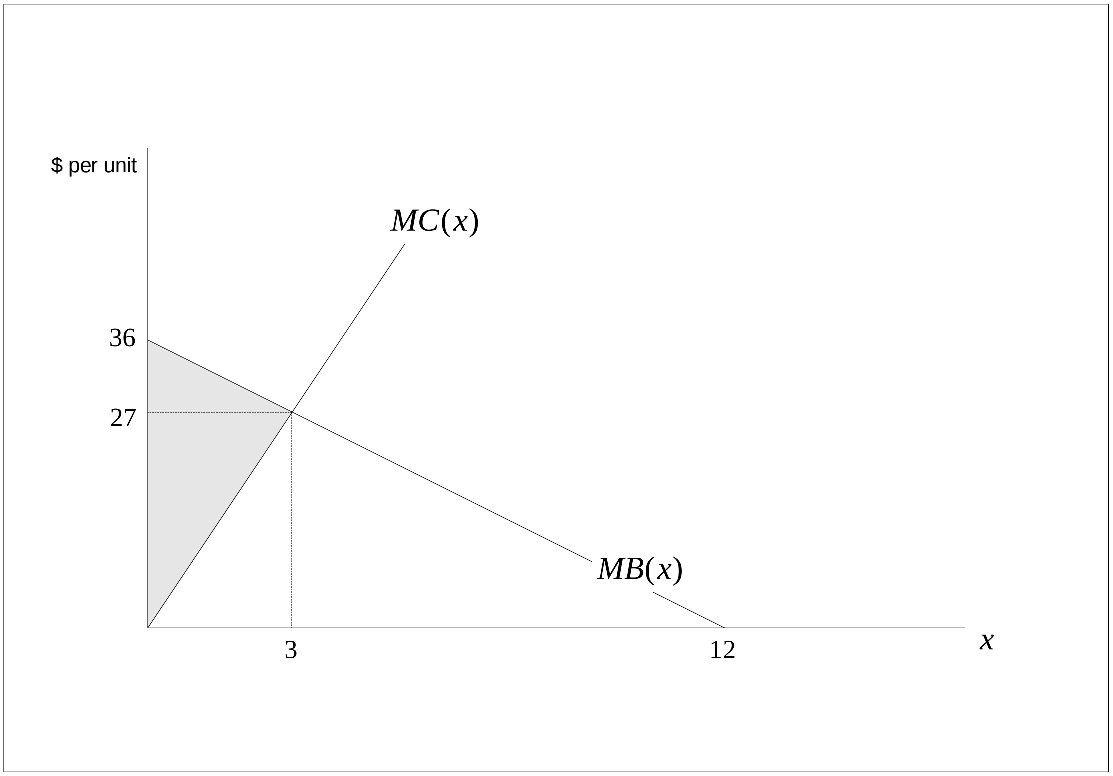


FIGURE RA-10

25. **D**. The rate of change of the cost function is the marginal cost function. Evaluated at $\hat{x} = 3$, it is equal to 27.

26. **D**. $W(x) = MB(x) - MC(x) = (36 - 3x) - (9x) = 36 - 12x$. It is depicted in Figure RA-11.

27. **C**. See Figure RA-11.

28. **B**. The horizontal intercept of $W(x)$ occurs where $W(x) = 0$. Since $W(x) = MB(x) - MC(x)$, it follows that the intercept of $W(x)$ is at $\hat{x} = 3$, where $MB(x) = MC(x)$.

29. **A**. Recall the answer to Q.14 above. The same logic applies here. $\int_0^3 W(x)dx$ is the shaded area in Figure RA-12. Its area is $\frac{36(3)}{2} = 54$

30. **B**. Again, recall the reasoning in the answer to Q.14 above.

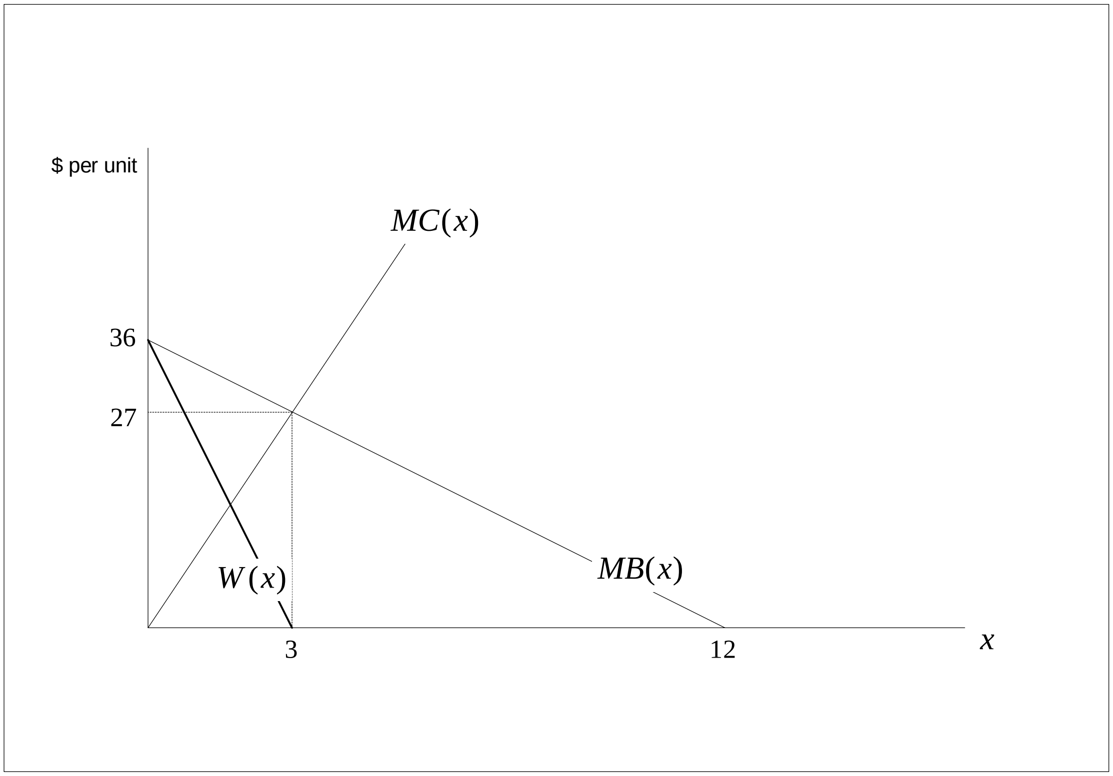


FIGURE RA-11

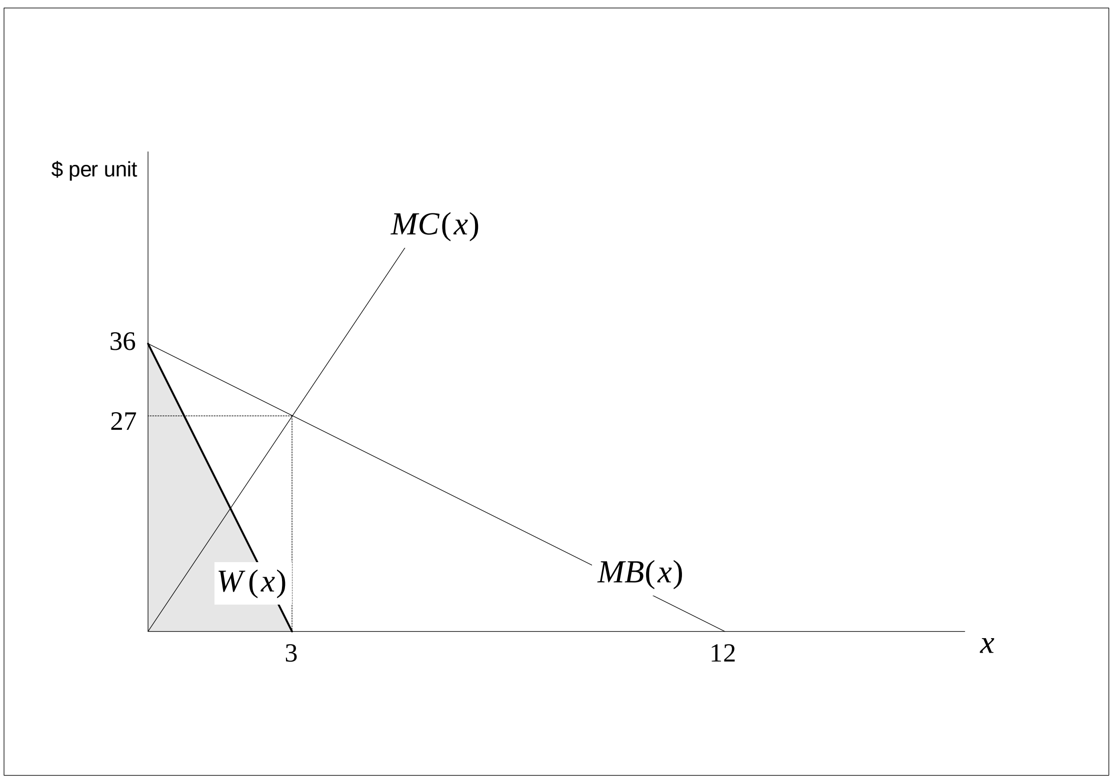


FIGURE RA-12

31. **D**. $S(x) = MC(x) + D(x) = 9x + 6x = 15x$. It is depicted in Figure RA-13.

32. **C**. The revised optimum is where $MB(x) = S(x)$. Setting $36 - 3x = 15x$ and solving for *x* yields $x^* = 2$. See Figure RA-14.

33. **C**. This is the shaded area in Figure RA-15. It comprises

$$A = \frac{6(2)}{2} \quad \text{plus} \quad B = 30(2)$$

34. **A**. This is the shaded area in Figure RA-16. It is equal to

$$\frac{30(2)}{2}$$

35. **A**. The lower area is

$$\int_0^2 D(x)dx$$

The upper area is the difference between the shaded area in Figure RA-16 and the shaded area in Figure RA-17:

$$\int_0^2 S(x)dx - \int_0^2 MC(x)dx = \int_0^2 [MC(x) + D(x)]dx - \int_0^2 MC(x)dx = \int_0^2 D(x)dx$$

36. **B**. This is the shaded area in Figure RA-18. It is equal to difference between the areas in Figure RA-15 and Figure RA-16.

37. **C**. In words, this is equal to

[area between 2 and 3 under $MB(x)$] – [area between 2 and 3 under $S(x)$]

Note that this difference is a negative value. It is the shaded area in Figure RA-19.

In absolute value, it is equal to

$$\frac{18(3-2)}{2} = 9$$

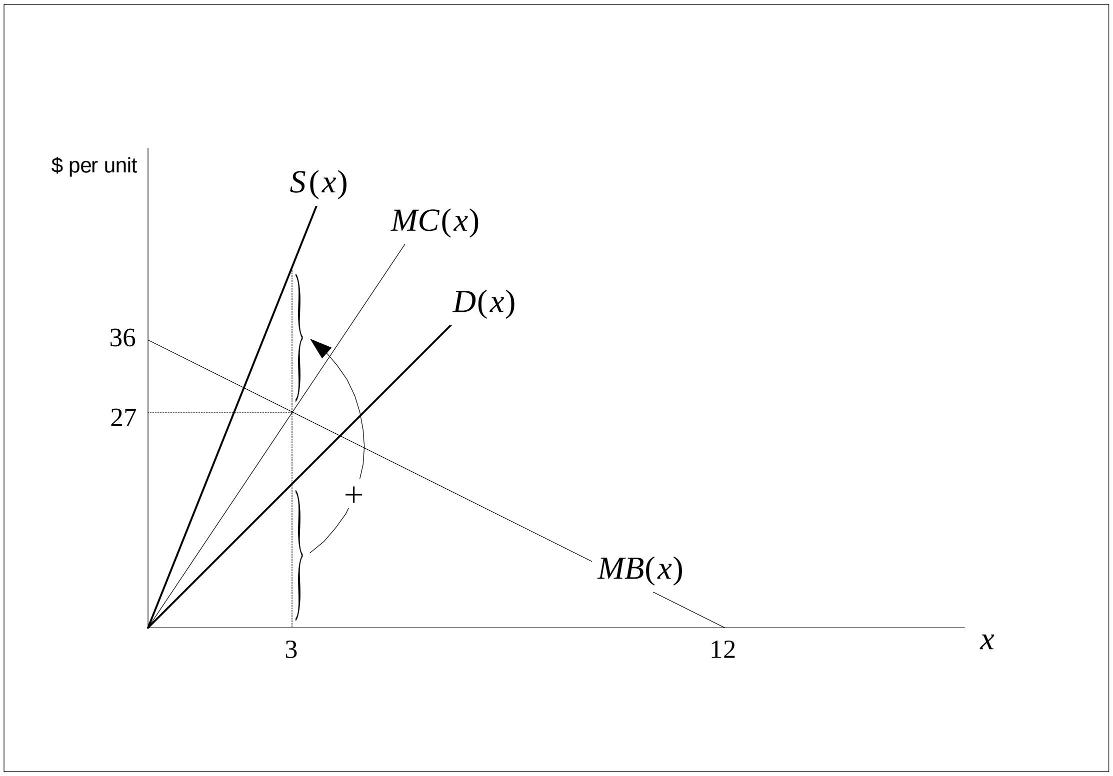


FIGURE RA-13

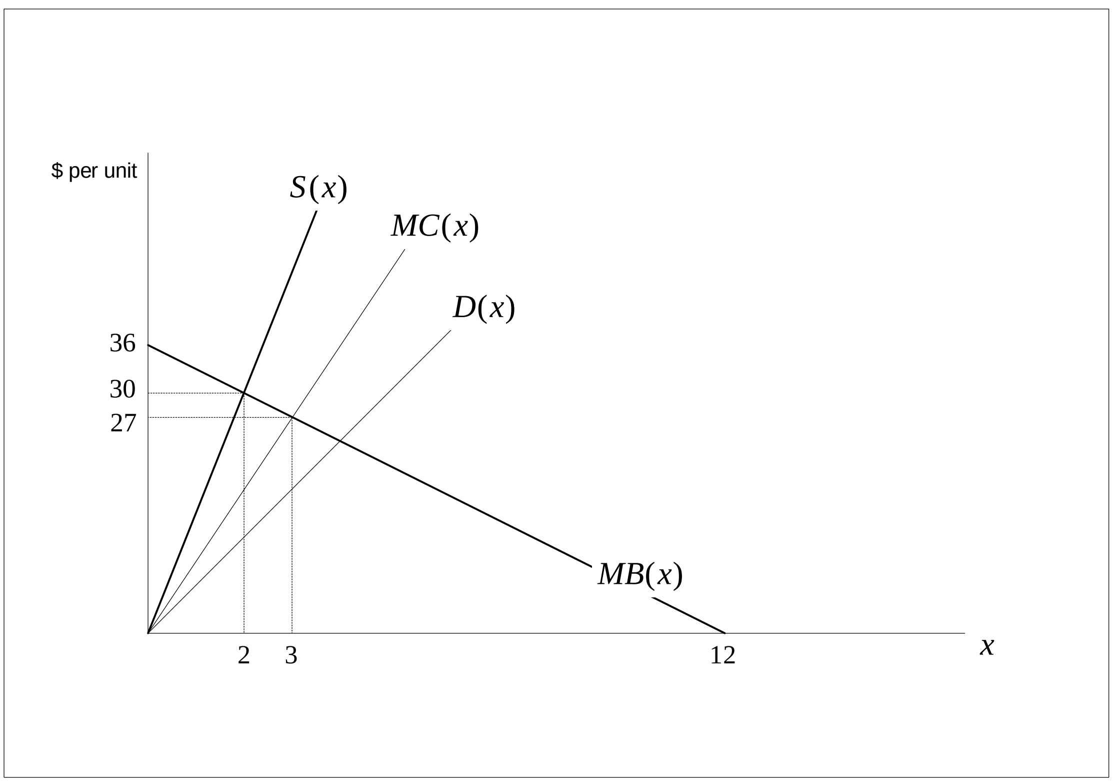


FIGURE RA-14

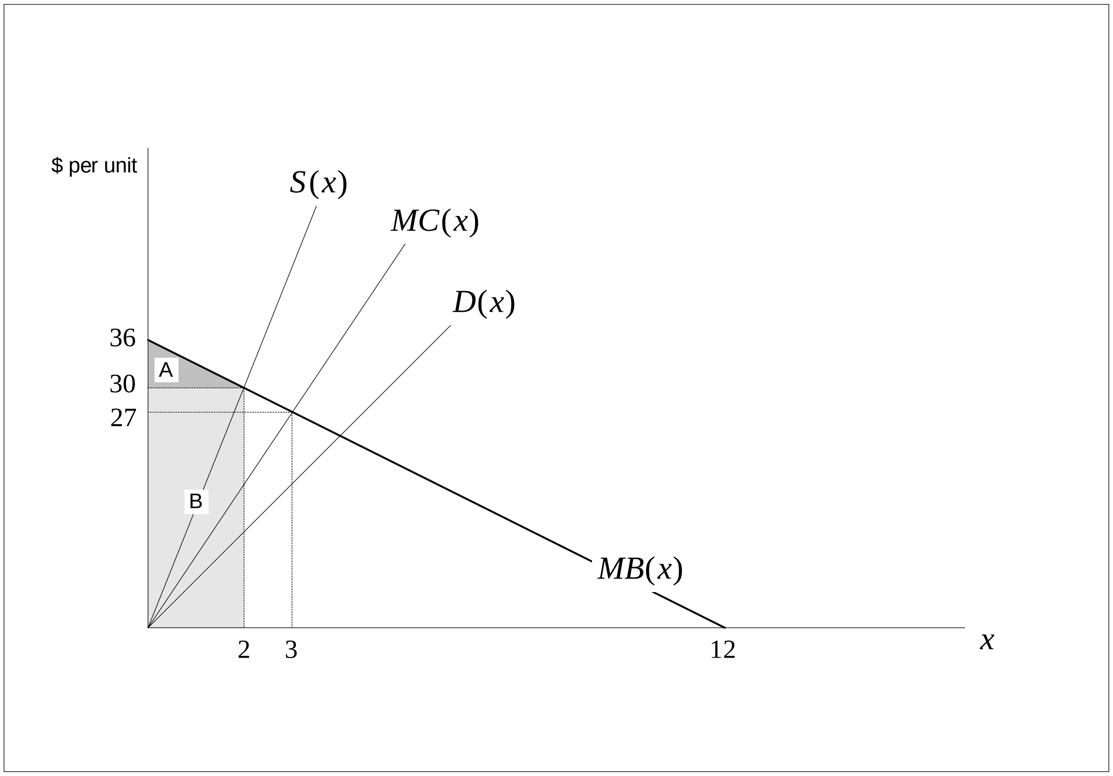


FIGURE RA-15

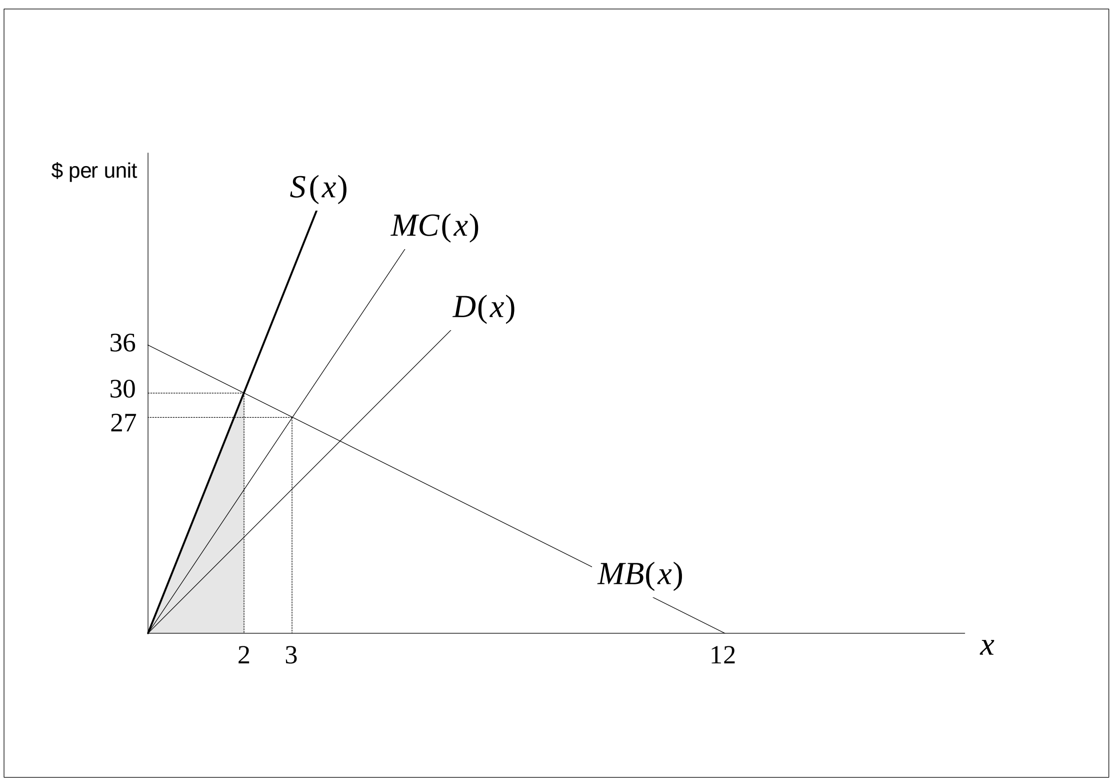


FIGURE RA-16

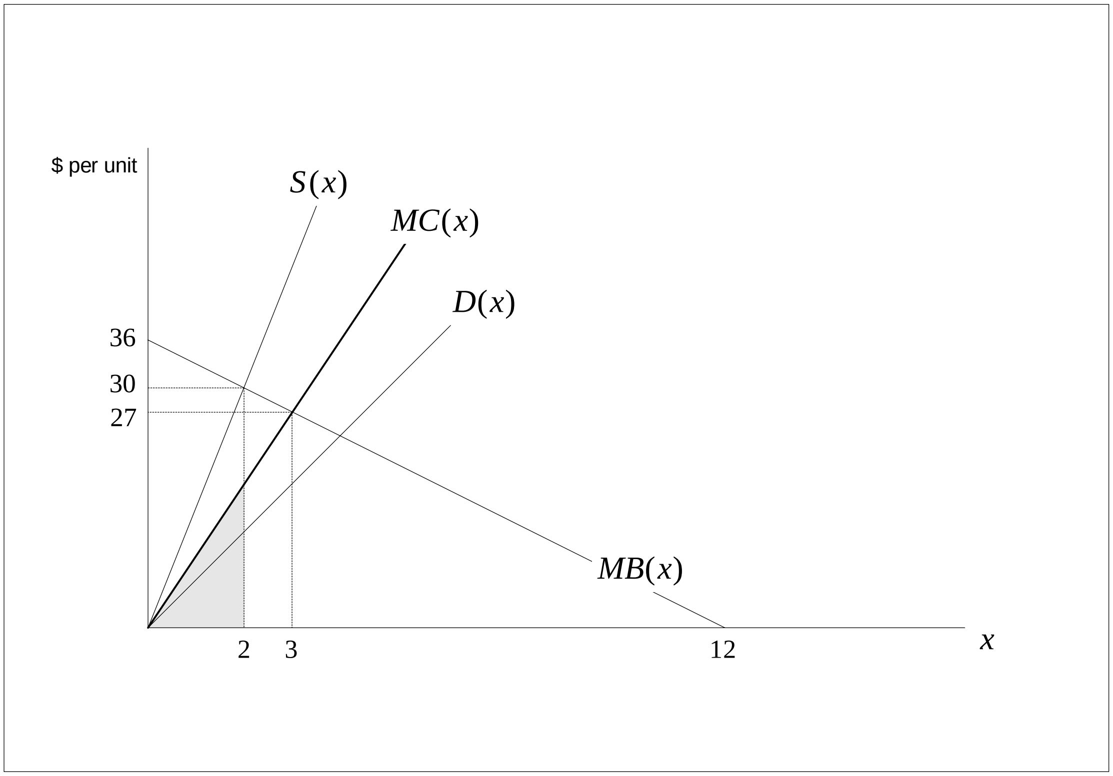


FIGURE RA-17

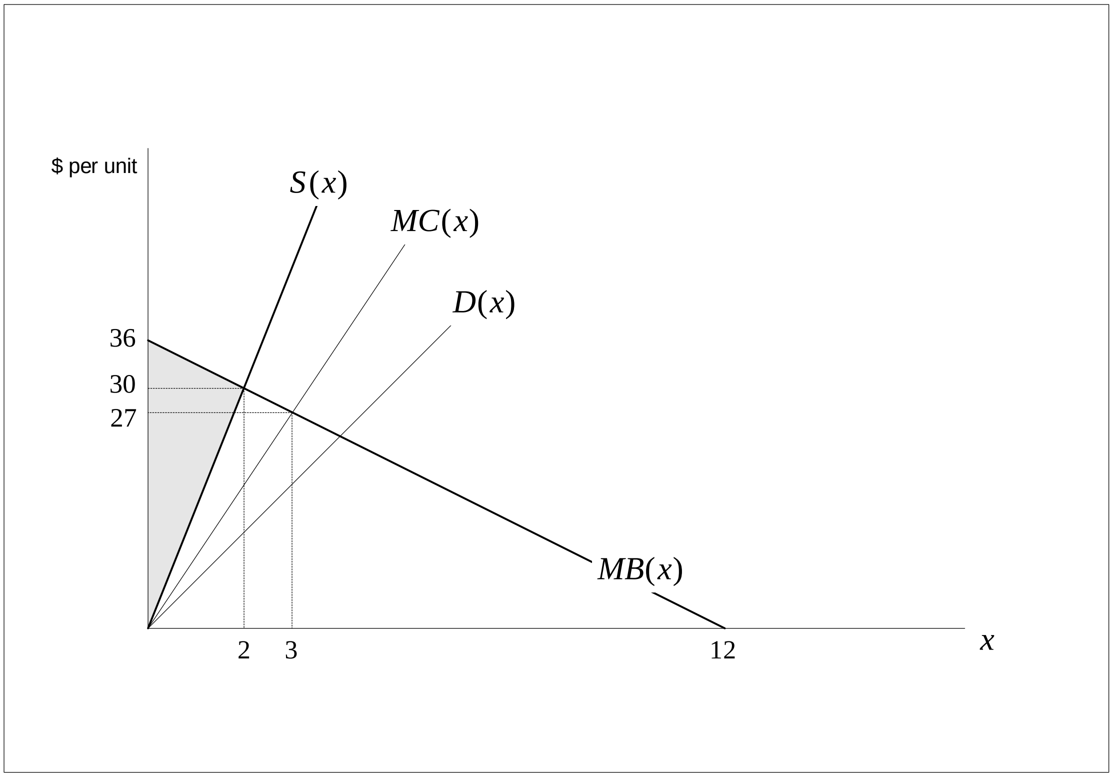


FIGURE RA-18

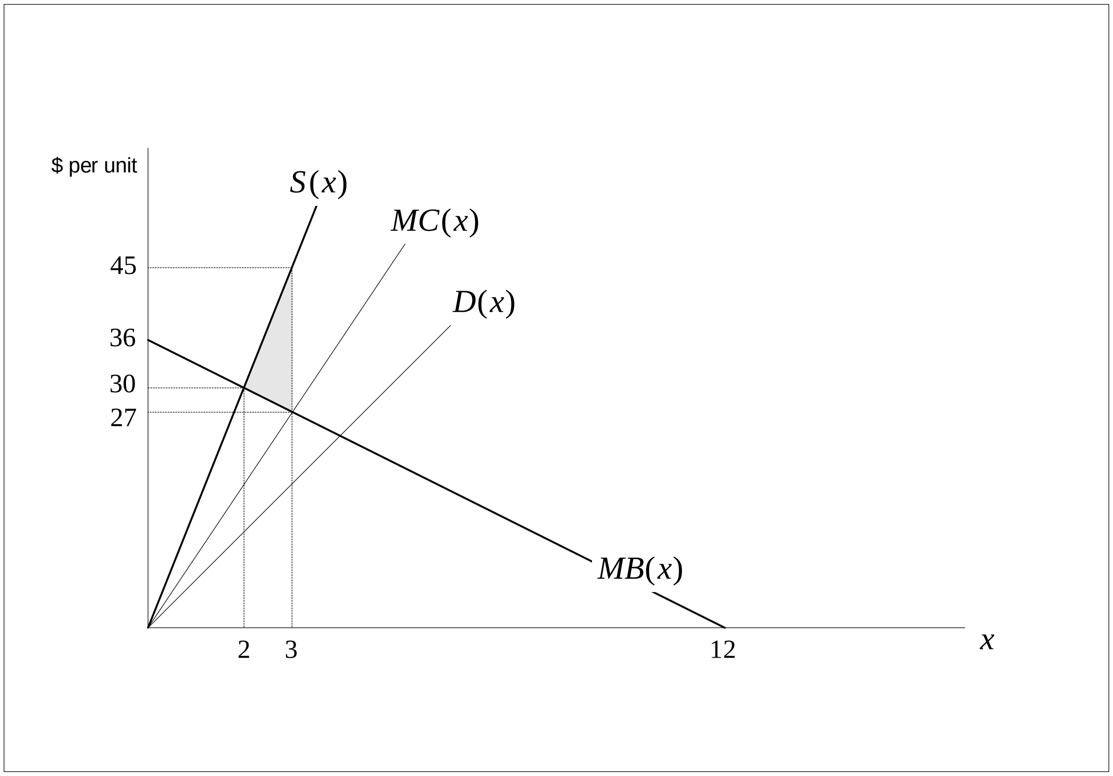


FIGURE RA-19

38. **C**. Inverting the inverse demand of person 1:

$$w = 40 - 2x_1 \;\rightarrow\; 2x_1 = 40 - w \;\rightarrow\; x_1(w) = 20 - \frac{w}{2}$$

39. **A**. Inverting the inverse demand of person 2:

$$w = 50 - 8x_2 \;\rightarrow\; 8x_2 = 50 - w \;\rightarrow\; x_2(w) = \frac{50}{8} - \frac{w}{8} = \frac{25}{4} - \frac{w}{8}$$

40. **B**. Construct the aggregate demand:

$$X(w) = x_1(w) + x_2(w) = \left(20 - \frac{w}{2}\right) + \left(\frac{25}{4} - \frac{w}{8}\right) = \frac{105}{4} - \frac{5w}{8}$$

Invert this to fund the inverse aggregate demand (see Figure RA-20):

$$X = \frac{105}{4} - \frac{5w}{8} \;\rightarrow\; \frac{5w}{8} = \frac{105}{4} - X \;\rightarrow\; w(X) = 42 - \frac{8X}{5}$$

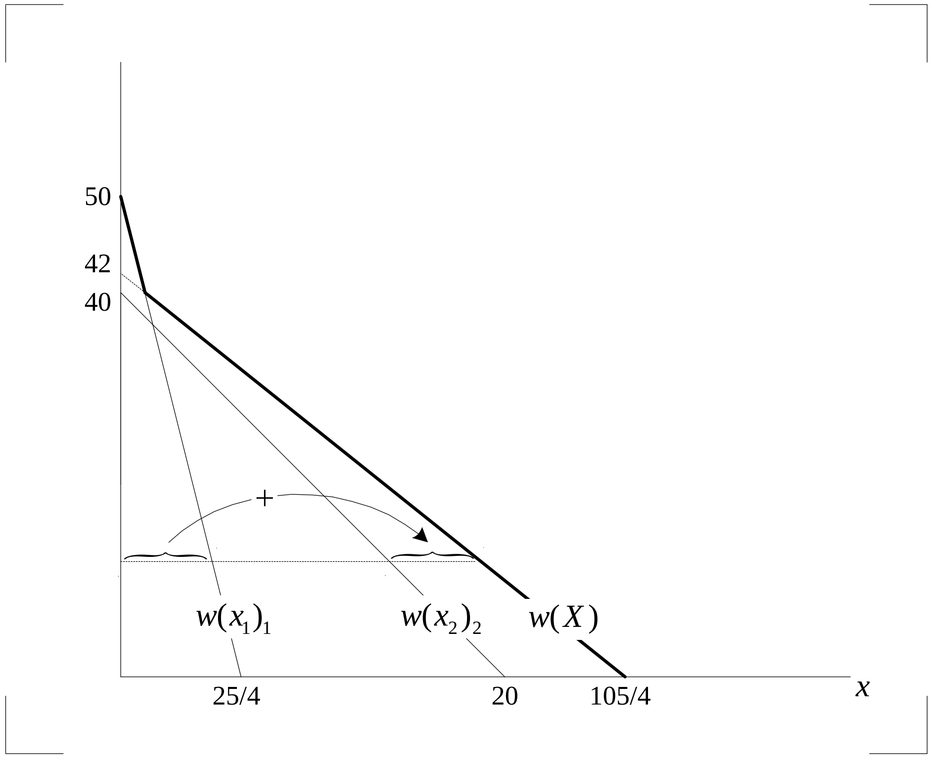


FIGURE RA-20